Block Cyclic SOR for Markov Chains With p-Cyclic Infinitesimal Generator

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ABSTRACT

We introduce a new application of p-cyclic iterations, for arbitrary $p \geqslant 2$. The block SOR method for the computation of the steady state-distribution of finite Markov chains that possess p-cyclic infinitesimal generators is considered. It is shown that convergence, in a sense more general than the usual, may be obtained even if the SOR iteration violates the usual conditions for semiconvergence. Necessary and

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sufficient conditions for convergence in this extended sense are derived. They are then applied in the case where the pth power of the associated Jacobi matrix of the system to be solved possesses only nonnegative eigenvalues. Exact convergence intervals and the optimal ω-values are derived for this case. In addition to the "usual" optimal ω in the interval $(1, p/(p-1))$, other ω-values that yield convergence in the extended sense are found to achieve the same optimal convergence rate. Numerical tests indicate that small perturbations of ω around the optimal value affect the convergence factor much less if these newly introduced optimal ω-values are used.

1. INTRODUCTION

Block iterative methods are particularly suitable for the solution of large and sparse systems of linear equations having matrices that possess a special structure. Here we consider block SOR. Given the system of equations

$$Ax = \mathbf{b}, \qquad A \in \Re^{n \times n}, \quad \mathbf{x}, \mathbf{b} \in \Re^{n}, \tag{1}$$

and the usual block decomposition

$$A = D - L - U, \tag{2}$$

where D, L, and U are block diagonal, lower triangular, and upper triangular matrices respectively and D is nonsingular, the block SOR method for any $\omega \neq 0$ is defined as

$$D\mathbf{x}^{(m)} = D\mathbf{x}^{(m-1)} + \omega\left(L\mathbf{x}^{(m)} - D\mathbf{x}^{(m-1)} + U\mathbf{x}^{(m-1)} + \mathbf{b}\right), \qquad m = 1, 2, \dots . \tag{3}$$

The method can be equivalently described as

$$\mathbf{x}^{(m)} = \mathscr{L}_\omega \mathbf{x}^{(m-1)} + \mathbf{c}, \qquad m = 1, 2, \dots, \tag{4}$$

where

$$\mathscr{L}_\omega = (D - \omega L)^{-1}[(1-\omega)D + \omega U] \tag{5}$$

and

$$\mathbf{c} = \omega(D - \omega L)^{-1}\mathbf{b}. \tag{6}$$

It is well known that, for general nonsingular systems (1), SOR converges iff $\rho(\mathscr{L}_\omega) < 1$. The associated convergence factor is then $\rho(\mathscr{L}_\omega)$. Very little is known about how the parameter ω affects the convergence of general systems. For the special but important case of "matrices with property A," Young [21] was able to obtain the optimum ω-value ω_{opt}, and the result was generalized by Varga [18, 19] for systems that have an associated block Jacobi matrix J that is weakly cyclic of index p.

DEFINITION 1.1. The matrix J is *weakly cyclic of index* p if there exists a permutation matrix P such that PJP^T has the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & B_1 \\ B_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & B_p & 0 \end{pmatrix} \tag{7}$$

where the null diagonal blocks are square.

When J is already in the form (7), it is called *consistently ordered.* For such matrices Varga proved the important relationship

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p \tag{8}$$

between the eigenvalues μ of J and λ of $\mathscr{L}_\omega$. Assuming further that all eigenvalues of J^p satisfy $0 \leqslant \mu^p \leqslant \rho(J^p) < 1$, he showed that the optimum ω-value ω_{opt} is the unique positive solution of the equation

$$[\rho(J)\omega]^p = p^p(p-1)^{1-p}(\omega - 1) \tag{9}$$

in the interval $(1, p/(p-1))$. This ω_{opt} yields a convergence factor equal to

$$\rho(\mathscr{L}_\omega) = (p-1)(\omega_{\text{opt}} - 1). \tag{10}$$

Similar results have been obtained for particular cases where the eigenvalues of J^p are nonpositive, i.e. $-p/(p-2) < -\rho(J)^p \leqslant \mu^p \leqslant 0$ [3, 11, 12, 15].

Wild and Niethammer took a more generic approach in examining the same problem. They showed [20] that the convergence of SOR and the optimal convergence factor can be determined by examining if the eigenvalues of J lie inside the subset of the complex plane which is bounded by some hypocloidal curve. This approach permitted them to rederive the results on ω_{opt} mentioned above, as well as exact convergence intervals. The work discussed in this paper follows this same approach. Furthermore, Eiermann, Niethammer, and Ruttan [6] extended the work of Wild and Niethammer [20] by treating the case when J^p has both positive and negative eigenvalues. Also, Pierce, Hadjidimos, and Plemmons [14] showed that, for the cases where J^p contains only real eigenvalues having the same sign, a repartitioning of the matrix A that reduces the index p will improve the convergence factor, that is, a repartitioning that yields 2-cyclic matrices is always superior to the original partitioning with $p > 2$.

All mentioned results consider *nonsingular* systems of equations of the form (1). Hadjidimos [8] examined the singular case [$\det A = 0$ and $\mathbf{b} \in \mathscr{R}(A)$]. Under the assumptions that J is weakly cyclic of index p, that the eigenvalues of J^p are nonnegative with $\rho(J) = 1$, and that J has a simple unit eigenvalue, he proved, among other results, that ω_{opt} is the unique root of (9) (in the same interval as in the nonsingular case), where $\rho(J)$ has to be replaced by $\tilde{\rho}(J)$, the maximum of the moduli of the eigenvalues of J excluding those of modulus 1, viz. $\tilde{\rho}(J) \equiv \max\{|\lambda|;\ \lambda \in \sigma(J),\ |\lambda| \neq 1\}$.

We are interested in using block SOR to compute the stationary probability distribution of an irreducible Markov chain with infinitesimal generator that is p-cyclic. (The p-cyclic form is defined in (13).) That is, we are interested in solving the homogenous system of equations

$$\boldsymbol{\pi} Q = \mathbf{0} \tag{11}$$

subject to the normalizing condition

$$\|\boldsymbol{\pi}\|_1 = 1, \tag{12}$$

where we assume that Q has a p-cyclic normal form

$$\begin{pmatrix} Q_{1,1} & Q_{1,2} & 0 & \cdots & 0 & 0 \\ 0 & Q_{2,2} & Q_{2,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{p-1,p-1} & Q_{p-1,p} \\ Q_{p,1} & 0 & 0 & \cdots & 0 & Q_{p,p} \end{pmatrix}, \tag{13}$$

where the subblocks $Q_{i,i}$, $i = 1, \dots, p$, are square and nonsingular. We are motivated in this study by the fact that infinitesimal generators coming from models of closed queueing networks with blocking, even those containing feedback connections, have a p-cyclic form (13) with the diagonal blocks being diagonal matrices [17], and the parameter p may be arbitrarily large, depending on the configuration of the network. Following an idea of Courtois and Semal [5], we show that, although the parameter ω may be chosen in a way such that $\mathscr{L}_\omega$ violates the conditions for semiconvergence, SOR "converges" to a vector that has subvectors parallel to the corresponding subvectors of the stationary probability vector $\boldsymbol{\pi}$. It is straightforward to apply an aggregation procedure described in Section 7 to compute the actual weights that should multiply the subvectors that SOR has produced. We derive conditions for the convergence of SOR in this extended sense. We then apply these conditions to find the exact convergence intervals for the case where the pth power of the block Jacobi matrix J which is associated with the system (11) possesses nonnegative eigenvalues only. The optimal ω-values are determined as well. New ω-values that give convergence in the extended sense, along with the "usual" optimal ω in $(1, p/(p-1))$ (see [8]), are found to achieve the optimal convergence factor. Numerical tests indicate that perturbations of ω around the optimal value affect the convergence rate much less if these new optimal values are used.

The organization of the paper is as follows. In Section 2 we show that the Jacobi matrix J associated with the system (11), when the matrix Q is of the form (13), is similar to a cyclic stochastic matrix, and we explore the consequences of this for the eigensystem of J. In Section 3, by exploiting an idea of Gutknecht et al. [7] which has also been by Wild and Niethammer [20], we introduce a p-step iterative method which employs J and whose convergence is closely related to the SOR convergence. The relations of the eigensystems of the Jacobi matrix J, the iteration matrix of the p-step method, and $\mathscr{L}_\omega$ are explored, and these relations lead to the introduction of the new sense of convergence and the derivation of necessary and sufficient conditions for convergence of this kind. Section 4 then develops the mathematical tools that are needed in order to determine if the convergence conditions are satisfied for some particular ω-value. These tools are used in Section 5 to derive the exact convergence intervals for the case of J^p having only nonnegative eigenvalues. The same techniques can be used to deal with other assumptions about the spectrum of J^p. In Section 6 we use the same tools to find the optimal ω-values for the same use. Section 7 describes the aggregation procedure mentioned above. Section 8 discusses a class of example models, derived from open queueing network modeling, possessing p-cyclic infinitesimal generators. In the same section, we discuss numerical results obtained by the application of block SOR to solve such models. The insensitivity of the convergence factor around the newly introduced optimal values is also discussed. Finally, in Section 9, we summarize the findings of the paper and explore directions for future work.

2. p-CYCLIC MARKOV CHAINS

Assume an irreducible and aperiodic Markov chain that possesses an infinitesimal generator Q of the form (13). We point out that such Markov chains do exist for arbitrary $p \geqslant 2$. See Section 8 for a class of examples derived from queueing network modeling. Assume also the partition

$$Q = \hat{D} - \hat{L} - \hat{U}, \tag{14}$$

where $\hat{D}$, $\hat{L}$, and $\hat{U}$ are block diagonal, strictly lower triangular, and strictly upper triangular matrices respectively. Then, the matrix

$$\hat{J} = \hat{D}^{-1}(\hat{L} + \hat{U}) \tag{15}$$

is an irreducible cyclic stochastic matrix with period p. (We use the term "period" for "index" in the remainder of the paper.) Indeed, since $-Q$ is a singular M-matrix, it follows that $-\hat{D}_{ii}$, $i = 1, \dots, p$, are nonsingular M-matrices [2, p. 156, Theorem 4.16] and therefore $-\hat{D}_{ii}^{-1}$ exist and are strictly positive [2, p. 141, Theorem 2.7]. Since $-(\hat{L} + \hat{U})$ is nonnegative (as the nondiagonal part of an infinitesimal generator), we conclude that $\hat{J} = \hat{D}^{-1}(\hat{L} + \hat{U})$ is nonnegative. Furthermore, since Q is an infinitesimal generator, $Q\mathbf{e} = \mathbf{0}$, where $\mathbf{e}$ is a column vector of all ones. By Equations (14) and (15) we immediately get that

$$\hat{J}\mathbf{e} = \hat{D}^{-1}(\hat{L} + \hat{U})\mathbf{e} = \mathbf{e},$$

which, along with the nonnegativity, proves that $\hat{J}$ is a stochastic matrix. The irreducibility comes from the fact that $-Q$ is an irreducible M-matrix [2, Theorems 4.16 and 4.12]. Lastly, observe that, by construction $\hat{J}$ is cyclic. The period is p, for otherwise some of the blocks in $\hat{J}$ could be partitioned further, or equivalently Q would be k-cyclic for some $k > p$.

We now state the problem (11) in the form (1) as

$$Q^T\mathbf{x} = \mathbf{0}, \qquad \|\mathbf{x}\|_1 = 1, \tag{16}$$

where $\mathbf{x} = \boldsymbol{\pi}^T$. Let Q^T be partitioned as in (2). Obviously, referring to (14), $D = \hat{D}^T$, $L = \hat{U}^T$, and $U = \hat{L}^T$. The block Jacobi matrix associated with the system (16) is

$$J = D^{-1}(L + U). \tag{17}$$

The correspondence between the matrices into which $\hat{J}$ is partitioned and those of the partition of J gives

$$J = D^{-1}\hat{J}^T D.$$

We can now state

THEOREM 2.1. *The block Jacobi matrix* (17) *associated with the system* (16), *where* Q *is of the form* (13), *is similar to the transpose of an irreducible and periodic stochastic matrix with period* p.

Notice that J is, by construction, cyclic of the form (7).

By Theorem 2.1 and by application of Romanovski's theorem [19, Theorem 2.4] we conclude that the pth roots of unity

$$\beta_k = e^{i2k\pi/p}, \qquad k = 0, 1, \dots, p-1, \tag{18}$$

are eigenvalues of J with multiplicity one. Furthermore, all nonzero eigenvalues of J appear in p-tuples, each member of a tuple having the same multiplicity, that is, if $\mu \in \sigma(J)\setminus\{0\}$ then $\mu\beta_k \in \sigma(J)$, $k = 1, \dots, p-1$, with the same multiplicity as μ. We will call all $\mu\beta_k$, $k = 0, \dots, p-1$, eigenvalues of the same *cyclic class* of J.

The interrelation among the eigenvectors of an irreducible and cyclic stochastic matrix corresponding to eigenvalues of the same cyclic class was observed by Courtois and Semal, who proved a theorem, an immediate generalization of which we now state [5, Theorem 1]:

THEOREM 2.2. *Let* T^T *be of the form* (7), *similar to a cyclic stochastic matrix with period* p. *Let also* $\psi = (\psi_1, \dots, \psi_p)$ *be a left eigenvector associated with the eigenvalue* $\mu \neq 0$ *of* T, *where the partitioning of* ψ *into subvectors is conformal with the partition of* T *into blocks. Then, a corresponding left eigenvector of* T *associated with the eigenvalue* $\mu^{(k)} = \mu\beta_k$, $k = 1, \dots, p-1$, *where* β_k *is defined by* (18), *is* (*up to a single multilplicative constant*)

$$\psi^{(k)} = \left(\psi_1, \beta_k^{-1}\psi_2, \beta_k^{-2}\psi_3, \dots, \beta_k^{-(p-1)}\psi_p\right).$$

The (trivial) extension lies in the fact that Courtois and Semal were interested in proving the theorem only for the dominant eigenvalue $\mu = 1$. The same proof, however, remains valid for any $\mu \neq 0$. For the proof see [5].

Observe that, since J is of the form (7), by Theorem 2.1 we deduce the validity of Theorem 2.2 for the *right* eigenvectors of a cyclic class of J.

The fact that the subvectors of the eigenvectors corresponding to the eigenvalues within the same class remain unchanged will prove to be of crucial importance for the determination of the convergence of the SOR method. One more thing is also worth noting. While here we are examining the application of block SOR to solve continuous Markov chain problems of the form (11) [or the equivalent system (16)] with infinitesimal generator Q of the form (13), the results apply to discrete Markov chain problems with a transition probability matrix P cyclic with period p. Indeed, in this latter case, the problem to be solved is

$$\boldsymbol{\pi} P = \boldsymbol{\pi}, \qquad \|\boldsymbol{\pi}\|_1 = 1,$$

or, equivalently,

$$(I - P^T)\boldsymbol{\pi}^T = \mathbf{0}, \qquad \|\boldsymbol{\pi}\|_1 = 1.$$

It is immediate that, P being a cyclic stochastic matrix with transpose of the form (7), the corresponding homogeneous problem has a matrix that is of the form (13) and the associated Jacobi matrix is P^T. Therefore, all the results of the paper carry over, simply by replacing J with P^T.

3. THE p-STEP ITERATION AND CONDITIONS FOR SOR CONVERGENCE

The SOR iteration (4) was observed by Gutknecht (see Gutknecht, Niethammer, and Varga [7]) to be related to the following simpler iteration when the block Jacobi matrix that is associated with the system (1) has the form (7). We introduce the *p-step relaxation*

$$\mathbf{x}^{(m)} = \omega J \mathbf{x}^{(m-1)} + (1-\omega)\mathbf{x}^{(m-p)} + \omega \mathbf{c}', \qquad m \geqslant p, \tag{19}$$

where

$$J = D^{-1}(L+U)$$

is the Jacobi matrix associated with the system (1) with the splitting (2) imposed, and

$$\mathbf{c}' = D^{-1}\mathbf{b}. \tag{20}$$

Let now the iterates $\mathbf{x}^{(m)} = (\mathbf{x}_1^{(m)}, \dots, \mathbf{x}_p^{(m)})$ be partitioned conformally with the partition of J. (Here and below we use the notation for a vector and its transpose loosely.) By considering successive block equations arising from consecutive iterations of the p-step method, it is immediate that

$$\begin{aligned}
\mathbf{x}_1^{(m)} &= \omega B_1 \mathbf{x}_p^{(m-1)} + (1-\omega)\mathbf{x}_1^{(m-p)} + \omega \mathbf{c}'_1, \\
\mathbf{x}_2^{(m+1)} &= \omega B_2 \mathbf{x}_1^{(m)} + (1-\omega)\mathbf{x}_2^{(m-p+1)} + \omega \mathbf{c}'_2, \\
&\vdots \\
\mathbf{x}_p^{(m+p-1)} &= \omega B_p \mathbf{x}_{p-1}^{(m+p-2)} + (1-\omega)\mathbf{x}_p^{(m-1)} + \omega \mathbf{c}'_p.
\end{aligned} \tag{21}$$

If, on the other hand, we apply one SOR iteration to the vector

$$\mathbf{y}^{(m-1)} = \left(\mathbf{x}_1^{(m-p)}, \mathbf{x}_2^{(m-p+1)}, \dots, \mathbf{x}_p^{(m-1)}\right)^T,$$

we get

$$\mathbf{y}^{(m+p-1)} = \left(\mathbf{x}_1^{(m)}, \mathbf{x}_2^{(m+1)}, \dots, \mathbf{x}_p^{(m+p-1)}\right)^T,$$

that is, the block components, after one SOR step is applied to $\mathbf{y}^{(m-1)}$, are exactly the same as if a whole sweep (21) (involving p iterations) of the p-step relaxation (19) were applied. We therefore see that, in the case where $\mathbf{b}$ (and $\mathbf{c}'$) are nonvanishing, SOR converges exactly when the p-step iteration converges, and SOR is p times faster. This is not necessarily true in our case. The system (16) that we wish to solve is a homogeneous system with a coefficient matrix of order n with rank $n-1$. The solution of (16) is thus specified up to a multiplicative constant. The successive SOR iterates $\mathbf{y}^{(m-1)}$ and $\mathbf{y}^{(m+p-1)}$ that we used above were composed of subvectors coming from previous successive iterates of the p-step iteration. If these iterates carry different multiplicative constants, then SOR may converge to the true solution, while the p-step iteration may "converge" to a vector composed of subvectors parallel to the solution, but multiplied with different factors, so that the whole p-step solution is not parallel to the solution of (16). This behavior indicates signs of cyclicity of the p-step iteration that are not shared by SOR. We now proceed to make this discussion more concrete by examining the eigensystems of the iteration matrices associated with the p-step iteration and with SOR.

The p-step relaxation can be transformed to an equivalent first-order relaxation (see Gutknecht, Niethammer, and Varga [7])

$$\mathbf{r}^{(m)} = T\mathbf{r}^{(m-1)} + \omega\mathbf{c}'', \qquad m = p, p+1, \ldots, \tag{22}$$

where

$$\mathfrak{R}^{np} \ni \mathbf{r}(m) = \left(\mathbf{x}^{(m)}, \mathbf{x}^{(m-1)}, \ldots, \mathbf{x}^{(m-p+1)}\right)^T \tag{23}$$

with the subvectors $\mathbf{x}^{(i)}$ being employed in the iteration (19),

$$\mathfrak{R}^{np} \ni \mathbf{c}'' = \left(\mathbf{c}', \mathbf{0}, \ldots, \mathbf{0}\right)^T$$

with $\mathbf{c}'$ as in (20), and

$$\mathfrak{R}^{np,np} \ni T = \begin{pmatrix} \omega J & 0 & 0 & \cdots & 0 & (1-\omega)I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix} \tag{24}$$

with I being the identity matrix of dimension n. Since we are interested in solving thes homogeneous problem (16), both method (4) and (22) reduce to the power method applied to the relevant iteration matrix [i.e., the vectors $\mathbf{b}$ in (1), $\mathbf{c}$ in (4), and $\mathbf{c}''$ in (22) are all zero vectors]. In view of Theorem 2.2, we seek to relate the eigensystems of $\mathscr{L}_\omega$ and T with the eigensystem of J. The next two theorems perform this task.

THEOREM 3.1 (See also [20]). *The eigenvalues of T are the np zeros of the n polynomials $f_{\omega,\mu_j}(\lambda) \equiv \lambda^p - \omega\mu_j\lambda^{p-1} - (1-\omega)$, $j = 1, \ldots, n$, where μ_j, $j = 1, \ldots, n$, are the n eigenvalues of J. Furthermore, if $\lambda_{j,k}$ is the* kth *root of $f_{\omega,\mu_j}(\lambda)$, then the corresponding right eigenvector of T is (up to a single multiplicative constant)*

$$\left(\lambda_{j,k}^{p-1}\boldsymbol{\psi}_j, \lambda_{j,k}^{p-2}\boldsymbol{\psi}_j, \ldots, \lambda_{j,k}\boldsymbol{\psi}_j, \boldsymbol{\psi}_j\right)^T,$$

where $\boldsymbol{\psi}_j$ is a right eigenvector of J associated with the eigenvalue μ_j, i.e., $J\boldsymbol{\psi}_j = \mu_j \boldsymbol{\psi}_j$.

Proof. Let λ be an eigenvalue of T, and $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_p)^T$ be an associated eigenvector. Then

$$T\mathbf{z} = \lambda \mathbf{z},$$

or, by considering (24),

$$\omega J \mathbf{z}_1 + (1-\omega)\mathbf{z}_p = \lambda \mathbf{z}_1$$

and

$$\mathbf{z}_i = \lambda \mathbf{z}_{i+1}, \qquad i = 1, \dots, p-1.$$

We therefore get that

$$\mathbf{z}_i = \lambda^{p-i}\mathbf{z}_p, \qquad i = 1, \dots, p, \tag{25}$$

and that

$$\lambda^{p-1}\omega J \mathbf{z}_p + (1-\omega)\mathbf{z}_p = \lambda^p \mathbf{z}_p,$$

or

$$J\mathbf{z}_p = \frac{\lambda^p - (1-\omega)}{\omega\lambda^{p-1}}\mathbf{z}_p.$$

This last relation yields that $\mathbf{z}_p$ is a right eigenvector of J corresponding to some eigenvalue μ_j. Therefore,

$$\mu_j = \frac{\lambda^p - (1-\omega)}{\omega\lambda^{p-1}},$$

which immediately leads to $f_{\omega,\mu_j}(\lambda) = 0$. By using (25) we complete the proof. ■

The connection between the SOR iterations (4) and the p-step relaxation (22) is further illuminated by the next theorem.

THEOREM 3.2. *If λ is an eigenvalue of T, then λ^p is an eigenvalue of $\mathcal{L}_\omega$. Furthermore, if μ_j is the eigenvalue of J to which λ is related to by Theorem* 3.1, *then the right eigenvector of $\mathcal{L}_\omega$ corresponding to the eigenvalue λ^p is (up to a multiplicative constant) $\boldsymbol{\psi}' = (\boldsymbol{\psi}_1, \lambda\boldsymbol{\psi}_2, \dots, \lambda^{p-1}\boldsymbol{\psi}_p)^T$, where $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_p)^T$ is a right eigenvector of J corresponding to the eigenvalue μ_j and where the partition of vectors into subvectors is conformal with the partition of J into blocks.*

Proof. We just need to prove that

$$\mathcal{L}_\omega \boldsymbol{\psi}' = \lambda^p \boldsymbol{\psi}'.$$

Let, therefore,

$$\mathbf{y} = \mathcal{L}_\omega \boldsymbol{\psi}'.$$

By using (5), we get that

$$(D - \omega L)\mathbf{y} = \left[(1-\omega)D + \omega U\right]\boldsymbol{\psi}',$$

or

$$(I - \omega D^{-1}L)\mathbf{y} = \left[(1-\omega)I + \omega D^{-1}U\right]\boldsymbol{\psi}',$$

and by (17) and (7), it follows that

$$\mathbf{y}_1 = (1-\omega)\boldsymbol{\psi}_1 + \omega\lambda^{p-1}B_1\boldsymbol{\psi}_p \tag{26}$$

and

$$\mathbf{y}_i = \omega B_i \mathbf{y}_{i-1} + (1-\omega)\lambda^{i-1}\boldsymbol{\psi}_i, \qquad i = 2, \dots, p. \tag{27}$$

But since $J\boldsymbol{\psi} = \mu_j \boldsymbol{\psi}$, we have that

$$B_1 \boldsymbol{\psi}_p = \mu_j \boldsymbol{\psi}_1$$

and

$$B_i \boldsymbol{\psi}_{i-1} = \mu_j \boldsymbol{\psi}_i, \qquad i = 2, \dots, p,$$

so that (26) becomes

$$\mathbf{y}_1 = \left[(1-\omega) + \omega\lambda^{p-1}\mu_j\right]\boldsymbol{\psi}_1,$$

which, in view of $f_{\omega,\mu_j}(\lambda) = 0$, gives

$$\mathbf{y}_1 = \lambda^p \boldsymbol{\psi}_1 = \lambda^p \boldsymbol{\psi}_1'.$$

By using (27) we inductively get

$$\mathbf{y}_i = \lambda^{p+i-1}\boldsymbol{\psi}_i = \lambda^p \boldsymbol{\psi}_i', \qquad i = 2,\ldots,p,$$

which gives us the desired result. ∎

Theorem 3.2 seems to contradict the fact that there are np eigenvalues of T, while $\mathscr{L}_\omega$ is only of order n. Notice, however, that if Λ_{μ_j} is the set of the p eigenvalues of T corresponding to the eigenvalue μ_j of J, viz.

$$\Lambda_{\mu_j} \equiv \left\{\lambda \middle| f_{\omega,\mu_j}(\lambda) = 0\right\},$$

then the set of the p eigenvalues of T corresponding to some other eigenvalue of J in the same cyclic class with μ_j [namely $\mu_j\beta_k$, with β_k as in (18)] is

$$\Lambda_{\mu_j\beta_k} = \left\{\lambda \middle| f_{\omega,\mu_j\beta_k}(\lambda) = 0\right\} = \left\{\lambda\beta_k \middle| \lambda \in \Lambda_{\mu_j}\right\}, \qquad k = 1,\ldots,p-1.$$

It is now immediate that if $\lambda' \in \Lambda_{\mu_j}$ and $\lambda'' \in \Lambda_{\mu_j\beta_k}$ then $(\lambda')^p = (\lambda'')^p$. In other words, the p eigenvalues of J in the same cyclic class produce [via the roots of the polynomials $f_{\omega,\mu_j\beta_k}(\lambda)$, $k = 0,\ldots,p-1$] only p and not p^2 eigenvalues of $\mathscr{L}_\omega$. By using Theorems 2.2 and 3.2, one also sees that the corresponding eigenvectors coincide as they should.

The previous observations reveal that, just by examining the roots of $f_{\omega,\mu_j}(\lambda)$, we can determine the eigenvalues of $\mathscr{L}_\omega$ associated with *all* $\mu_j\beta_k$, where $k = 0,\ldots,p-1$. The roots of $f_{\omega,1}(\lambda)$ are of particular importance, since, as Theorems 3.2 and 2.2 dictate, the eigenvectors of the p eigenvalues of $\mathscr{L}_\omega$ that are equal to the pth powers of these roots will have subvectors parallel to the corresponding subvectors of the Perron eigenvector of J, which is the solution of the system (16) that we wish to compute. We therefore proceed to develop some notation that will allow us to describe the properties of these roots.

Let us define $\alpha(\omega)$ as

$$\alpha(\omega) = \max\{|\lambda| \,|\, f_{\omega,1}(\lambda) = 0\}, \tag{28}$$

that is, the maximum modulus of any eigenvalue of the p-step iteration matrix that is associated with the unit eigenvalue of J. Notice that $\alpha(\omega) \geqslant 1$ $\forall \omega$, since, trivially, $f_{\omega,1}(1) = 0$ $\forall \omega$.

Let also $\vartheta(\omega)$ be the maximum modulus among the moduli of all eigenvalues of T that correspond to all eigenvalues of J, excluding those eigenvalues of J that belong to the same cyclic class with the unit eigenvalue, viz.

$$\vartheta(\omega) = \max\left\{|\lambda| \,\middle|\, f_{\omega,\mu}(\lambda) = 0,\ \mu \in \sigma(J),\ |\mu| < 1\right\}. \tag{29}$$

Finally, let $\mathscr{A}(\omega)$ be

$$\mathscr{A}(\omega) = \{\lambda \,|\, f_{\omega,1}(\lambda) = 0,\ |\lambda| = \alpha(\omega)\}, \tag{30}$$

that is, the set of all eigenvalues of T which correspond to the unit eigenvalue of J and have the maximum modulus $\alpha(\omega)$. By the definition of $\alpha(\omega)$ in (28), it is immediate that $|\mathscr{A}(\omega)|$, the cardinality of $\mathscr{A}(\omega)$, is greater than or equal to one.

Recall now that, taking into account the fact that $1 \in \sigma(\mathscr{L}_\omega)$ $\forall_\omega$, the usual conditions for semiconvergence (see e.g. [2]) become:

A1. $\rho(\mathscr{L}_\omega) = 1$.

A2. All elementary divisors associated with the unit eigenvalue are linear, i.e., $\operatorname{rank}(I - \mathscr{L}_\omega) = \operatorname{rank}(I - \mathscr{L}_\omega)^2$.

A3. If $\lambda \in \sigma(\mathscr{L}_\omega)$ with $|\lambda| = 1$, then $\lambda = 1$.

In the light of the previous definitions these become:

B1. $\alpha(\omega) = 1$ and $\vartheta(\omega) < \alpha(\omega)$.

B2. 1 is a simple root of $f_{\omega,1}(\lambda) = 0$.

B3. $|\mathscr{A}(\omega)| = 1$.

Condition B1 is equivalent to condition A1. Since the eigenvalue 1 of $\mathscr{L}_\omega$ will be associated with a single eigenvector (by Theorems 3.2 and 2.2), B2 is equivalent to A2. Finally, condition B3, along with the second part of B1, guarantees that there is no other eigenvalue of $\mathscr{L}_\omega$ having modulus one. It is therefore equivalent to A3.

It is evident now that the SOR method will converge to the solution of the system (16) for exactly those ω values that satisfy conditions B1–B3. Consider, however, the following set of (more relaxed) constraints:

C1. $\vartheta(\omega) < \alpha(\omega)$.

C2. If $\lambda \in \mathscr{A}(\omega)$, then λ is a simple root of $f_{\omega,1}(\lambda) = 0$, or else it is a multiple root, but the principle vectors associated with the Jordan block of $\mathscr{L}_\omega$ corresponding to the eigenvalue λ^p have subvectors parallel to the corresponding subvectors of the Perron eigenvector of J.

Obviously, the set of values for the parameter ω that satisfies conditions B1–B3 is a subset of the set that satisfies C1–C2. Assume now that, for some specific ω, conditions C1–C2 are satisfied while B1–B3 are not. Then, there are some (maybe more than one) eigenvalues of $\mathscr{L}_\omega$ that have modulus $\alpha(\omega)^p \geqslant 1$, the maximum among the moduli of all eigenvalues of $\mathscr{L}_\omega$. Since condition C2 is satisfied, either all such eigenvalues have multiplicity one, or the corresponding principle vectors (remember, there is only one eigenvector associated with λ^p) have subvectors parallel to the desired solution. Therefore, SOR will converge to a linear combination of the eigenvectors (or eigenvectors and principle vectors) associated with these eigenvalues. Theorem 3.2, in conjunction with Theorem 2.2, assures that this linear combination will maintain subvectors that are parallel to the corresponding subvectors of the stationary probability vector. If we apply the simple aggregation step that is described in Section 7, we can retrieve the desired solution.

It is easy to see that conditions C1–C2 are *necessary* for the subvectors to be parallel to the subvectors of the solution of (16). We postpone until the next section the proof that condition C2 holds for any $\omega \in \mathscr{R} \setminus \{0\}$ and is therefore redundant. Hence, we can state

THEOREM 3.3. *The SOR method converges to a vector that has subvectors parallel to the corresponding subvectors of the solution of the system* (16) *if and only if the parameter* ω *is chosen such that condition* C1 *is satisfied. The partition of all vectors into subvectors is meant to be conformal with the partition of the corresponding block Jacobi matrix* J *into blocks. In case of convergence, the associated convergence factor is* $[\vartheta(\omega)/\alpha(\omega)]^p$. *The SOR method converges, in the standard sense, if and only if conditions* B1–B3 *are satisfied.*

We wish to stress that, for an ω-value that satisfies conditions C1–C2 but does not satisfy B1–B3, SOR does *not* converge in the standard sense. The SOR relaxations converge to a vector that is *different* from the solution of

(16). It happens, though, that the desired solution can be trivially obtained, given the vector to which SOR converges. In this sense, the method converges to the desired solution. Any reference to convergence in the sequel, unless explicitly stated otherwise, is to be understood as convergence with this extended sense.

Note also that since we relaxed condition B1 to condition C1, we do not need to insist on $\rho(\mathscr{L}_\omega)=1$. In particular, we do not have to impose the restriction $0<\omega<2$. Any value of $\omega \in \Re$ that satisfies conditions C1–C2 will assure convergence. However, the restriction $\omega \neq 0$ remains, for otherwise SOR degenerates to a null iteration. In the following sections we are going to study the convergence of the SOR method in the light of conditions C1–C2. Cases where the more restrictive conditions B1–B3 are satisfied (and SOR converges in the standard sense) will be pointed out. The value $\omega=2$ will, however, be excluded from this study. This is because, as is easy to see, the roots of $f_{1,\mu}(\lambda)$ are μ with multiplicity one and 0 with multiplicity $p-1$.

It is therefore immediate that the block Gauss-Seidel converges (in the standard sense) and the associated convergence factor is $\tilde{\rho}(J)^p$, where $\tilde{\rho}(J)$ is the maximum modulus among the moduli of the eigenvalues of J, excluding the eigenvalues belonging to the same cyclic class with the unit eigenvalue. It is worth noting that, as Theorem 3.2 dictates, the Gauss-Seidel iteration matrix is not cyclic, although the Jacobi matrix is cyclic.

4. TOOLS FOR INVESTIGATING THE CONVERGENCE OF SOR

As became apparent in Section 3, the SOR relaxations converge exactly when conditions C1–C2 are satisfied. Here we build tools that will help determine the validity of these conditions for different values of the parameter ω. It turns out that condition C2 holds for all real ω-values (besides the exceptional value 0). The following two theorems determine the quantity $\alpha(\omega)$ for different ω-ranges.

THEOREM 4.1. *If* $\omega>0$, *then*:

(a) *If* $\omega \leqslant p/(p-1)$, *then* $\alpha(\omega)=1$ *and* $\mathscr{A}(\omega)=\{1\}$. *If* $\omega<p/(p-1)$, *then* $f_{\omega,1}(\lambda)$ *has* 1 *as a simple root. If* $p=p/(p-1)$, *then* 1 *is a double root.*

(b) *If* $\omega>p/(p-1)$, *then* $\alpha(\omega)=\gamma(\omega)>1$ *and* $\mathscr{A}(\omega)=\{\gamma(\omega)\}$, *where* $\gamma(\omega)$ *is the unique simple positive root of* $f_{\omega,1}(\lambda)$ *in the interval*

$(\omega(p-1)/p, \omega)$. *Furthermore, the relation*

$$\left(\frac{p-1}{p}\omega\right)^{p} > (p-1)(\omega-1) \tag{31}$$

holds.

The proof is in Appendix A.

Notice that when $\omega = p/(p-1)$, the dominant root 1 is double and it is not immediate that condition C2 holds. Later on in this section, it is shown that condition C2 is satisfied for this ω-value as well. Note also that if we want to restrict ourselves to convergence in the usual sense, only values of ω in the range $(0, p/(p-1))$ are candidates, something consistent with the already known results (e.g. [8]). The next theorem investigates negative values of ω.

THEOREM 4.2. *If $\omega < 0$, then:*

(a) *If p is even, then $\alpha(\omega) = -\gamma(\omega) > 1$ and $\mathscr{A}(\omega) = \{\gamma(\omega)\}$, where $\gamma(\omega)$ is the unique simple negative root of $f_{\omega,1}(\lambda)$ in $[\omega-1, -(1-\omega)^{1/p})$.*

(b) *If p is odd, then let ω^* be the unique negative root of*

$$g(x) = \left(x\frac{p-1}{p}\right)^{p} - (p-1)(x-1) \tag{32}$$

in $[-p, -p/(p-1))$. Then:

(i) *If $\omega < \omega^* < -p/(p-1)$, then $\alpha(\omega) = -\gamma(\omega) > 1$ and $\mathscr{A}(\omega) = \{\gamma(\omega)\}$, where $\gamma(\omega)$ is the unique simple negative root of $f_{\omega,1}(\lambda)$ in the interval $(\omega, ((p-1)/p)\omega)$. The relation*

$$\left(\frac{p-1}{p}\omega\right)^{p} < (p-1)(\omega-1) \tag{33}$$

holds.

(ii) *If $\omega = \omega^*$, then $\gamma(\omega^*) = ((p-1)/p)\omega^*$ and the root $\gamma(\omega^*)$ is double.*

(iii) *If $\omega > \omega^*$, then let t^* be the unique solution of*

$$U_{p-1}^{p}(t) = -\frac{\omega^{p}}{1-\omega} U_{p-2}^{p-1}(t) \tag{34}$$

in $(\cos(\pi/p),1)$, *where* $U_{p-1}(t)$ *and* $U_{p-2}(t)$ *are the Chebyshev polynomials of the second kind, of degrees* $p-1$ *and* $p-2$ *respectively. Let also*

$$\gamma^{p}(\omega) = -(1-\omega)U_{p-2}(t^{*}) < -(1-\omega). \tag{35}$$

Then $\alpha(\omega) = |\gamma(\omega)|$,

$$\mathscr{A}(\omega) = \left\{\gamma(\omega)e^{j\cos^{-1}t^{*}}, -\gamma(\omega)e^{j\cos^{-1}(-t^{*})}\right\},$$

and the roots in $\mathscr{A}(\omega)$ *are simple. Furthermore, the relation*

$$\left(\frac{p-1}{p}\omega\right)^{p} > (p-1)(\omega-1) \tag{36}$$

holds.

The proof is in Appendix B. Notice that, since for $\omega < 0$ we always have $\alpha(\omega) > 1$, no negative values for the parameter ω may be candidates for convergence in the standard sense.

From Theorems 4.1 and 4.2, it is immediate that condition C2 of Section 3 is satisfied by all real values for the parameter ω, except for $\omega = p/(p-1)$ and, if p is odd, for $\omega = \omega^{*}$, with ω^{*} as in Theorem 4.2. These two ω-values make $f_{\omega,1}(\lambda)$ have a double dominant root. Since there is a single eigenvector associated with the eigenvalue 1 [$\gamma^{p}(\omega^{*})$ for $\omega = \omega^{*}$] of $\mathscr{L}_{\omega}$, there is a Jordan block of size 2 associated with this eigenvalue. The next theorem assures us that the principal vector associated with this eigenvalue has subvectors parallel to the solution vector. Thus, condition C2 is still satisfied, and convergence in the extended sense is guaranteed for these ω-values as well.

THEOREM 4.3. *When the block SOR is employed on the system* (16) *with* ω *equal to* $p/(p-1)$ *or* (*for odd* p) *to* ω^{*}, *the principal vector associated with the Jordan block* (*of size* 2) *of the dominant eigenvalue of* $\mathscr{L}_{\omega}$ *has subvectors parallel to the corresponding subvectors of the Perron eigenvector of the associated block Jacobi matrix* J.

The proof is in Appendix C.

Having determined the parameter $\alpha(\omega)$ as a function of ω, we now describe a tool that will be of help in deciding about the validity of condition C1 for some given ω. In this task we follow Wild and Niethammer [20]. For condition C1 to be valid, for some given ω, we require that all eigenvalues of

T [the iteration matrix of the p-step method defined in (24)], excluding those related to the eigenvalues of J having modulus unity, have modulus strictly less than $\alpha(\omega)$. Define $\tilde{\sigma}(J)$ as

$$\tilde{\sigma}(J) = \sigma(J) \setminus \{\beta_k | k = 0, \ldots, p-1\}$$

with β_k as in (18). Also let $\tilde{\rho}(J)$ be

$$\tilde{\rho}(J) = \max\{|\lambda| \,|\, \lambda \in \tilde{\sigma}(J)\} = \max\{|\lambda| \,|\, \lambda \in \sigma(J), |\lambda| < 1\}. \tag{37}$$

The above requirement for C1's condition validity can now be expressed as

$$\text{if} \quad |\lambda'| \geqslant \alpha(\omega) \quad \text{then} \quad f_{\omega,\mu}(\lambda') \neq 0, \quad \mu \in \tilde{\sigma}(J).$$

This becomes

$$1 - (1-\omega)\frac{1}{\lambda'^p} \neq \omega\mu\frac{1}{\lambda'}, \quad |\lambda'| \geqslant \alpha(\omega), \qquad \mu \in \tilde{\sigma}(J),$$

or, by setting $\lambda = 1/\lambda'$,

$$\mu \neq \frac{1-(1-\omega)\lambda^p}{\omega\lambda} \equiv q_\omega(\lambda), \qquad |\lambda| \leqslant \frac{1}{\alpha(\omega)}, \qquad \mu \in \tilde{\sigma}(J). \tag{38}$$

If we define $\overline{D}_{1/\alpha(\omega)}$ as the closed disk, centered at the origin, with radius $1/\alpha(\omega)$, and if $q_\omega(\overline{D}_{1/\alpha(\omega)})$ is the mapping of that disk through the function $q_\omega(\lambda)$, then condition C1 is satisfied iff

$$\tilde{\sigma}(J) \subset U_{1/\alpha(\omega)}(\omega) \equiv \mathscr{C} \setminus q_\omega\big(\overline{D}_{1/\alpha(\omega)}\big).$$

Similarly, if we define D_η as the open disk with radius $\eta > 1/\alpha(\omega)$, and $U_\eta(\omega)$ as

$$U_\eta(\omega) = \mathscr{C} \setminus q_\omega(D_\eta),$$

then, if $\tilde{\sigma}(J) \subset U_\eta(\omega)$, the p-step iteration converges with a convergence factor better or equal to $1/\alpha(\omega)\eta$, and SOR converges with a factor better or equal to $(1/\alpha(\omega)\eta)^p$. The convergence factor is exactly equal to $1/\alpha(\omega)\eta$ if there is at least one eigenvalue in $\tilde{\sigma}(J)$ on the boundary of $U_\eta(\omega)$ [in which case $\vartheta(\omega)$ is exactly $1/\eta$].

We now proceed to describe the boundary of $U_\eta(\omega)$ in general. Then the domain $U_\eta(\omega)$ will be the closed [open if $\eta = 1/\alpha(\omega)$] part of the complex plane that is bounded by the bounding curve and includes the point (0,0). If we set $\lambda = \eta e^{j\theta}$, $0 \leqslant \theta < 2\pi$, then the boundary of $U_\eta(\omega)$ is formed by the set of points

$$\{z | z \in \mathscr{C}, z = q_\omega(\eta e^{j\theta})\}$$

with $q_\omega(\lambda)$ as in (38). We therefore have

$$z = x + jy = \frac{1-(1-\omega)\eta^p e^{jp\theta}}{\omega\eta e^{j\theta}}. \tag{39}$$

Consider the case $\omega > 1$. From Equation (39), we get the following parametric equations for z:

$$x = \frac{1}{\omega\eta}\cos\theta + \frac{\omega-1}{\omega}\eta^{p-1}\cos(p-1)\theta \tag{40}$$

and

$$y = -\frac{1}{\omega\eta}\sin\theta + \frac{\omega-1}{\omega}\eta^{p-1}\sin(p-1)\theta. \tag{41}$$

Comparing with the parametric form of a hypocycloidal curve

$$x = (R-r)\cos\theta + h\cos\left(\frac{R}{r-1}\right)\theta \tag{42}$$

and

$$y = -(R-r)\sin\theta + h\sin\left(\frac{R}{r-1}\right)\theta, \tag{43}$$

we see that the equations (40) and (41) reveal that the boundary of $U_\eta(\omega)$

belongs to the class of hypocycloidal curves for $\omega > 1$. We just have to put

$$R = \frac{p}{(p-1)\omega\eta}, \qquad r = \frac{1}{(p-1)\omega\eta}, \qquad h = \frac{\omega - 1}{\omega}\eta^{p-1}. \tag{44}$$

We notice that the ratio R/r is always equal to p, meaning that the curve always consists of p congruent arcs or, equivalently, it coincides with itself when subjected to a rotation by an angle $2k\pi/p$, $k \in \mathcal{J}$.

In general, if $h < r$, then the curve is a *shortened* hypocycloid; if $h = r$, an *ordinary-cusped* one; and if $h > r$, a *stretched* one. In this last case, the parts of the complex plane inside the loops are not parts of $U_\eta(\omega)$. Furthermore, still talking of the stretched case, if h becomes greater than $R - r$, the set $U_\eta(\omega)$ becomes empty. The points on the curve with the maximum modulus occur at angles $\theta = 2k\pi/p$, $k = 0, \dots, p-1$, and the corresponding modulus is

$$d_{\max} = R - r + h, \tag{45}$$

and the points with minimum modulus occur at angles $\theta = [(2k+1)/p]\pi$, $k = 0, \dots, p-1$, with corresponding modulus

$$d_{\min} = R - r - h. \tag{46}$$

A more detailed geometric characterization can be found in Wild and Niethammer [20]. See Figures 1–3 for examples of shortened cusped and stretched hypocycloids.

Having dealt with the properties of the hypocycloidal curves, which occur as boundaries of the set $U_\eta(\omega)$ for $\omega > 1$, we now investigate the boundaries

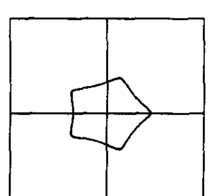

FIG. 1. Shortened nonrotated hypocycloid.

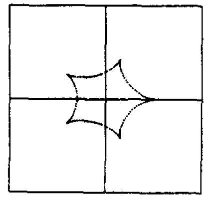

FIG. 2. Cusped nonrotated hypocycloid.

for the other ranges of ω. When $0 < \omega < 1$, by multiplying (39) by $e^{j\pi/p}$, we get

$$z' = ze^{j\pi/p} = \frac{1-(1-\omega)\eta^p e^{jp\theta}}{\omega\eta e^{j\theta}} e^{j\pi/p}$$

$$= \frac{1+(1-\omega)\eta^p e^{jp(\theta-\pi/p)}}{\omega\eta e^{j(\theta-\pi/p)}},$$

and by setting $\theta' \equiv \theta - \pi/p$, we see that the curve is again within the class of hypocycloidal curves, rotated counterclockwise by an angle π/p. By comparing with the equations (42) and (43), one easily gets

$$R = \frac{p}{\omega\eta(p-1)}, \qquad r = \frac{1}{\omega\eta(p-1)}, \qquad h = \frac{1-\omega}{\omega}\eta^{p-1}. \tag{47}$$

See Figures 4–6 for examples of rotated hypocycloids. For negative ω-val-

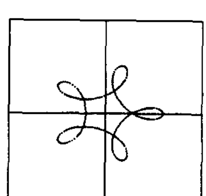

FIG. 3. Stretched nonrotated hypocycloid.

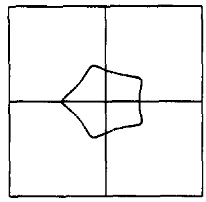

FIG. 4. Shortened rotated hypocycloid.

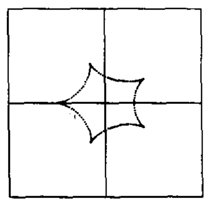

FIG. 5. Cusped rotated hypocycloid.

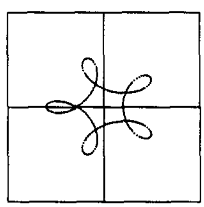

FIG. 6. Stretched rotated hypocycloid.

ues, if p is even, we again rotate the curve by an angle π/p; recalling that $e^{p\pi}=1$, we get

$$z' = ze^{j\pi/p} = \frac{1-(1-\omega)\eta^p e^{jp\theta}}{\omega\eta ej\theta} e^{j\pi/p}$$

$$= \frac{1+(1-\omega)\eta^p e^{jp(\theta+\pi-\pi/p)}}{-\omega\eta e^{j(\theta+\pi-\pi/p)}};$$

and, setting $\theta' \equiv \theta + \pi - \pi/p$, we see that we get a hypocycloid rotated counterclockwise by an angle of π/p, with parameters

$$R = -\frac{p}{\omega\eta(p-1)}, \qquad r = -\frac{1}{\omega\eta(p-1)}, \qquad h = -\frac{1-\omega}{\omega}\eta^{p-1}. \quad (48)$$

Finally for p odd (and $\omega < 0$) we recall that $e^{jp\pi} = -1$, and from (39) we get

$$z = \frac{1-(1-\omega)\eta^p e^{jp\theta}}{\omega\eta e^{j\theta}} = \frac{1+(1-\omega)\eta^p e^{jp(\theta+\pi)}}{-\omega\eta e^{j(\theta+\pi)}},$$

and by setting $\theta' \equiv \theta + \pi$, we determine that the boundary of $U_\eta(\omega)$ in this case is a *non*rotated hypocycloid with parameters given, again, by (48).

The tools developed in this section will be used in the next sections to find the exact intervals for the parameter ω that yield convergence and to determine the optimal ω-value and the associated convergence factor.

5. CONVERGENCE RESULTS

We are now ready to investigate the convergence of the SOR method. We are making the assumption that all eigenvalues of J^p are nonnegative reals. This implies that

$$\tilde{\sigma}(J) \subset \left\{\tilde{\rho}(J)e^{j2k\pi/p} \middle| 0 < \tilde{\rho}(J) < 1,\ k = 0,\dots,p-1\right\}$$

with $\tilde{\sigma}(J)$ and $\tilde{\rho}(J)$ as defined in Section 4. As we saw in the same section, the domains $U_\eta(\omega)$ are symmetric under rotations by angles $2k\pi/p$, $k \in \mathcal{Z}$, so we just need to investigate if the interval $(0,1)$ is contained in $U_{1/\alpha(\omega)}(\omega)$ to decide about the convergence of the SOR method.

Consider first the case of $0 < \omega < 1$. The boundary of $U_{1/\alpha(\omega)}(\omega)$ is a hypocycloid, rotated by π/p. The domain $U_\eta(\omega)$ is nonempty iff $R - r - h > 0$, or, by Equation (47),

$$R - r - h = \frac{1}{\omega\eta} - \frac{1-\omega}{\omega}\eta^{p-1} > 0,$$

which is equivalent to

$$\frac{1}{\eta^{p}} > 1 - \omega.$$

Since $\eta = 1/\alpha(\omega)$, by Theorem 4.1(a) the previous relation is equivalent to $1 > 1 - \omega$, which is certainly true.

Observe that since the hypocycloid is rotated, the curve meets the real positive semiaxis at a point with abscissa $d_{\min}$, given by (46). This value is

$$d_{\min} = R - r - h = \frac{1}{\omega} - \frac{1-\omega}{\omega} = 1,$$

where we took into account that $\eta = 1/\alpha(\omega) = 1$. We therefore conclude that the interval $(0,1)$ is included in $U_{1/\alpha(\omega)}(\omega)$; hence SOR converges in the interval $0 < \omega < 1$.

Proceeding to the case $1 < \omega \leqslant p/(p-1)$, we have for the boundary of $U_{1/\alpha(\omega)}(\omega)$ a *non*rotated hypocycloid. The parameters are given by (44). The hypocycloid is shortened (ordinary) [and $U_{1/\alpha(\omega)}(\omega)$ nonempty] iff $h < r(h+r)$. From Equation (44), we see that $h \leqslant r$ is equivalent to

$$\frac{\omega-1}{\omega}\eta^{p-1} \leqslant \frac{1}{(p-1)\omega\eta},$$

or

$$\frac{1}{\eta^{p}} \geqslant (p-1)(\omega-1),$$

and by recalling that $\eta = 1/\alpha(\omega)$, we get that $U_{1/\alpha(\omega)}(\omega)$ is nonempty iff

$$\alpha^{p}(\omega) \geqslant (p-1)(\omega-1). \tag{49}$$

Theorem 4.1(a) makes this last relation equivalent to

$$1 \geqslant (p-1)(\omega-1).$$

But since $\omega \leqslant p/(p-1)$, the above relation is certainly true. Having established the fact that the domain $U_{1/\alpha(\omega)}(\omega)$ is nonempty, we notice that the bounding hypocycloid cuts the real positive semiaxis at a point with abscissa $d_{\max}$, which By equation (44) is seen to be

$$d_{\max} = \frac{1}{\omega} + \frac{\omega - 1}{\omega} = 1,$$

where we have used the fact that $\eta = 1/\alpha(\omega) = 1$. Again, the interval $(0, 1)$ is contained inside $U_{1/\alpha(\omega)}(\omega)$, and SOR converges.

We next examine the case $\omega > p/(p-1)$. The bounding hypocycloid is as in the previous case. The domain $U_{1/\alpha(\omega)}(\omega)$ is still nonempty iff (49) holds. By Theorem 4.1(b) we see that $\alpha(\omega) > [(p-1)/p]\omega$ and that (31) holds. These two facts immediately yield that the inequality (49) is satisfied; equivalently, $U_{1/\alpha(\omega)}(\omega)$ is nonempty. The point at which the curve cuts the real positive semiaxis is

$$d_{\max} = \frac{\alpha(\omega)}{\omega} + \frac{\omega - 1}{\omega \alpha^{p-1}(\omega)} = \frac{\alpha^{p}(\omega) + \omega - 1}{\omega \alpha^{p-1}(\omega)}.$$

Again, Theorem 4.1(b) implies that $\alpha(\omega) = \gamma(\omega)$ and $f_{\omega,1}(\gamma(\omega)) = 0$, which immediately yields $d_{\max} = 1$. This assures the convergence of SOR in this case also.

It remains to investigate the case of negative ω-values. If p is even, we have a rotated (by an angle π/p) hypocycloid that bounds $U_{1/\alpha(\omega)}(\omega)$. Again, the set $U_{1/\alpha(\omega)}(\omega)$ is nonempty iff $R - r - h > 0$, which by Equation (48) becomes

$$-\frac{1}{\omega\eta} + \frac{1-\omega}{\omega}\eta^{p-1} > 0,$$

or, equivalently,

$$\alpha(\omega) = \frac{1}{\eta} > (1-\omega)^{1/p}.$$

By Theorem 4.2(a), we see that this holds. Furthermore, the curve cuts the real positive semiaxis at the point

$$d_{\min} = -\frac{p}{\omega\eta(p-1)} + \frac{1}{\omega\eta(p-1)} + \frac{1-\omega}{\omega}\eta^{p-1},$$

and after setting $\eta = 1/\alpha(\omega)$ and doing some algebraic manipulation, we get

$$d_{\min} = \frac{\alpha^p(\omega) - (1-\omega)}{-\omega\alpha^{p-1}(\omega)}.$$

Theorem 4.2(a) suggests that $\alpha(\omega) = -\gamma(\omega)$, and by recalling that p is even we get

$$d_{\min} = \frac{\gamma^p(\omega) - (1-\omega)}{\omega\gamma^{p-1}(\omega)}.$$

Since $f_{\omega,1}(\gamma(\omega)) = 0$, this gives $d_{\min} = 1$. Therefore, $(0,1) \subset U_{1/\alpha(\omega)}(\omega)$ and SOR converges.

Consider now the case of $\omega < 0$ and p odd. The boundary of $U_{1/\alpha(\omega)}(\omega)$ is a nonrotated hypocycloid. The hypocycloid is shortened (ordinary) [and $U_{1/\alpha(\omega)}(\omega)$ nonempty] iff $h \leqslant r$ or, by Equation (48), iff

$$-\frac{1-\omega}{\omega}\eta^{p-1} \leqslant -\frac{1}{\omega\eta(p-1)}$$

or, equivalently, iff

$$\alpha^p(\omega) = \frac{1}{\eta^p} \geqslant (p-1)(1-\omega).$$

By Theorem 4.2(b)(i), we see that if $\omega \leqslant \omega^*$, then since

$$\alpha(\omega) = -\gamma(\omega) \geqslant -\frac{p-1}{p}\omega,$$

we get

$$\alpha^p(\omega) \geqslant \left(-\frac{p-1}{p}\omega\right)^p = -\left(\frac{p-1}{p}\omega\right)^p$$

(with equality exactly for $\omega = \omega^*$) because p is odd, and, by (33) [or (32) for

$\omega = \omega^*$], we see that the hypocycloid is indeed shortened. It cuts the real positive semiaxis at

$$d_{\max} = R - r + h$$

which, by use of (48) becomes

$$d_{\max} = -\frac{1}{\omega\eta} - \frac{1-\omega}{\omega}\eta^{p-1} = -\frac{\alpha(\omega)}{\omega} - \frac{1-\omega}{\omega\alpha^{p-1}(\omega)}$$

or

$$d_{\max} = \frac{-\alpha^p(\omega) + (1-\omega)}{\omega\alpha^{p-1}(\omega)}.$$

By Theorem 4.2(b)(i), we get $\alpha(\omega) = -\gamma(\omega)$, and by recalling that p is odd, the previous equation becomes

$$d_{\max} = \frac{\gamma^p(\omega) - (1-\omega)}{\omega\gamma^{p-1}(\omega)}.$$

Since $f_{\omega,1}(\gamma(\omega)) = 0$, we conclude that $d_{\max} = 1$ and SOR converges.

Lastly, consider the case $\omega^* < \omega < 0$ (when p is odd of course). Now Theorem 4.2(b)(iii) applies. The hypocycloid that bounds $U_{1/\alpha(\omega)}(\omega)$ is, of course, as in the previous case. Since, $\alpha^p(\omega) = |\gamma^p(\omega)| = (1-\omega)U_{p-2}(t^*)$ and $t^* \in (\cos(\pi/p), 1)$, we have that $1 < U_{p-2}(t^*) < p-1$. Therefore,

$$(1-\omega) < \alpha^p(\omega) < (p-1)(1-\omega) \tag{50}$$

and the hypocycloid is stretched. We must verify that $R - r - h > 0$; otherwise $U_{1/\alpha(\omega)}(\omega)$ will be empty and SOR will not converge. From (48) it is easily seen that the condition $R - r - h$ is equivalent, in this case, to

$$-\frac{1}{\omega\eta} + \frac{1-\omega}{\omega}\eta^{p-1} > 0,$$

or equivalently, to

$$\alpha^p(\omega) = \frac{1}{\eta^p} > 1 - \omega,$$

which, from (50), is seen to be true. We have concluded that $U_{1/\alpha(\omega)}(\omega)$ is nonempty. Notice, however, that it is a *stretched* hypocycloid. As was commented in Section 4, the points inside and on the loops do not belong to $U_{1/\alpha(\omega)}(\omega)$, so we cannot proceed as usual, taking the point $d_{\max}$ as the rightmost point of the real positive semiaxis included in $U_{1/\alpha(\omega)}(\omega)$. We have to find the second (inner) point of intersection. From Equations (42), (43), and (48), we get

$$x = -\frac{1}{\omega\eta}\cos\theta - \frac{1-\omega}{\omega}\eta^{p-1}\cos(p-1)\theta$$

and

$$y = \frac{1}{\omega\eta}\sin\theta - \frac{1-\omega}{\omega}\eta^{p-1}\sin(p-1)\theta.$$

We must find the angle $\theta' \in (0, \pi/p)$ for which $y(\theta') = 0$. Equivalently, we must have

$$\frac{1}{\eta^p} = (1-\omega)\frac{\sin(p-1)\theta'}{\sin\theta'}.$$

If we set $t' = \cos\theta'$ and recall that $\alpha(\omega) = 1/\eta$, it becomes apparent that we must find the (unique) value $t' \in (\cos(\pi/p), 1)$ that satisfies

$$\alpha^p(\omega) = (1-\omega)U_{p-2}(t'). \tag{51}$$

Then the cutpoint at the real positive semiaxis will be

$$x = -\frac{\alpha(\omega)}{\omega}t' - \frac{1-\omega}{\omega\alpha^{p-1}(\omega)}T_{p-1}(t'),$$

which becomes

$$x = -\frac{\alpha(\omega)}{\omega}\left[t' + \frac{1-\omega}{\alpha^p(\omega)}T_{p-1}(t')\right].$$

If we now use (51), we get

$$x = -\frac{\alpha(\omega)}{\omega}\left[\frac{t'U_{p-2}(t') + T_{p-1}(t')}{U_{p-2}(t')}\right].$$

By an elementary property of Chebyshev polynomials (see [16, p. 9]), the numerator in the previous relation is equal to $U_{p-1}(t')$, so the relation becomes

$$x = -\frac{\alpha(\omega)}{\omega}\frac{U_{p-1}(t')}{U_{p-2}(t')}.$$

It is now immediate that

$$x^p = -\frac{\alpha^p(\omega)}{\omega^p}\frac{U^p_{p-1}(t')}{U^p_{p-2}(t')},$$

and, by using (51) again,

$$x^p = -\frac{1-\omega}{\omega^p}\frac{U^p_{p-1}(t')}{U^{p-1}_{p-2}(t')}. \tag{52}$$

Observe that Theorem 4.2(b)(iii) implies that t^*, as defined there, is the solution of (51); therefore, (52) is immediately seen to be equivalent to

$$x^p = 1.$$

We thus conclude, in this case also, $(0,1) \subset U_{1/\alpha(\omega)}(\omega)$, and SOR converges.

We can now summarize the results of this section in the following

THEOREM 5.1. *If the eigenvalues of J^p are nonnegative, then SOR converges for all $\omega \in \mathfrak{R} \setminus \{0\}$. Convergence in the standard sense is obtained for all $\omega \in (0, p/(p-1))$.*

The ω-range that gives standard semiconvergence is consistent with the results in [8].

6. THE OPTIMAL CONVERGENCE FACTOR AND THE ASSOCIATED ω-VALUES

In this section we determine the ω-values that yield the optimal convergence factor for block SOR when all the eigenvalues of J^p are nonnegative reals. We leave out the case where $\tilde{\rho}(J)=0$ [with $\tilde{\rho}(J)$ as defined in (37)], since, by the discussion at the end of Section 3, it is evident that the optimal choice in this case is $\omega=0$ (Gauss-Seidel), which yields a zero convergence factor (infinite convergence rate).

We first seek the ω-values for which the boundary of $U_\eta(\omega)$ is a cusped (ordinary) hypocycloid that meets the real positive semiaxis at $\tilde{\rho}(J)$. From Section 4 we know that we have a nonrotated hypocycloid for $\omega \in(-\infty,0)\cup(1,+\infty)$ when p is odd and for $\omega\in(1,+\infty)$ when p is even. For the case of $\omega>1$ the parameters of the hypocycloid are given by (44). It follows that the boundary of $U_\eta(\omega)$ is cusped $(r=h)$ iff

$$\frac{1}{\eta^p(\omega)}=\frac{1}{\tilde{\eta}^p(\omega)}=(p-1)(\omega-1).$$

Then the hypocycloid cuts the real positive semiaxis at

$$x=d_{\max}=\frac{1}{\omega\tilde{\eta}(\omega)}+\frac{\omega-1}{\omega}\tilde{\eta}^{p-1}(\omega)=\frac{p/(p-1)}{\omega\tilde{\eta}(\omega)}$$

We require that $x=\tilde{\rho}(J)$. After some algebraic manipulation, we get

$$h(\omega)\equiv\left(\frac{p-1}{p}\tilde{\rho}(J)\omega\right)^p-(p-1)(\omega-1)=0. \tag{53}$$

By using easy analytic arguments, we can see that the polynomial $h(\omega)$ has only two positive real roots: ω_0 in the interval

$$\left(1,\frac{p}{p-1}\right),$$

and ω_+ in the interval

$$\left(\frac{p}{p-1}\tilde{\rho}(J)^{-p/(p-1)}, \frac{p^{p/(p-1)}}{p-1}\tilde{\rho}(J)^{-p/(p-1)}\right).$$

The root ω_+, if used as the value for the parameter ω, will only yield convergence in the extended sense introduced in this paper.

We now turn to the case of $\omega < 0$ when p is odd. The parameters of the hypocycloid are now given by (48). We get an ordinary hypocycloid iff

$$\frac{1}{\eta^p(\omega)} = \frac{1}{\tilde{\eta}^p(\omega)} = (p-1)(1-\omega),$$

and the cutpoint at the real positive semiaxis is

$$x = d_{\max} = -\frac{1}{\omega\tilde{\eta}(\omega)} - \frac{1-\omega}{\omega}\tilde{\eta}^{p-1}(\omega) = -\frac{p/(p-1)}{\omega\tilde{\eta}(\omega)}.$$

We require again that $x = \tilde{\rho}(J)$, and after some algebra we get the relation (53) again. It can be easily seen that (for p odd only, the case we are examining here) $h(\omega)$ has a unique negative root ω_- in the interval

$$\left(-\frac{p^2}{p-1}\tilde{\rho}(J)^{-p/(p-1)}, -\frac{p}{p-1}\tilde{\rho}(J)^{-p/(p-1)}\right).$$

Notice that, with $g(x)$ as defined in Theorem 4.2, and since $h(\omega_-) = 0$, we have

$$\begin{aligned} g(\omega_-) &= \left(\frac{p-1}{p}\omega_-\right)^p - (p-1)(\omega_- - 1) \\ &= \left(\frac{p-1}{p}\omega_-\right)^p - \left(\frac{p-1}{p}\tilde{\rho}(J)\omega_-\right)^p \\ &= \left(\frac{p-1}{p}\omega_-\right)^p \left[1 - \tilde{\rho}(J)^p\right] < 0. \end{aligned}$$

The reasoning in the proof of Theorem 4.2 (see Appendix B) now reveals that $\omega_- < \omega^*$, with ω^* as defined in the same theorem.

Having found all the ω-values that yield nonrotated cusped hypocycloids with a cusp at $\tilde{\rho}(J)$, namely the values ω_0, ω_+, and (for odd p) ω_-, we will now show that these ω-values yield the same convergence factor, which is optimal. From the discussion in Section 4, we know that for ω equal to ω_0, ω_+, or ω_-, SOR converges with convergence factors r_0^p, r_+^p, and r_-^p respectively, where

$$r_0 = \frac{1}{\tilde{\eta}_0}, \qquad r_+ = \frac{1}{\tilde{\eta}_+ \alpha(\omega_+)}, \qquad r_- = \frac{1}{\tilde{\eta}_- \alpha(\omega_-)}$$

with $\alpha(\omega_+)$ and $\alpha(\omega_-)$ as in Theorem 4.1(b) and Theorem 4.2(b)(i) respectively, and with

$$\tilde{\eta}(\omega_0) \equiv \tilde{\eta}_0, \qquad \tilde{\eta}(\omega_+) \equiv \tilde{\eta}_+, \qquad \tilde{\eta}(\omega_-) \equiv \tilde{\eta}_-.$$

In the case of $\omega = \omega_0$, Theorem 4.1(a) guarantees that $\alpha(\omega_0) = 1$. We therefore have that

$$\frac{r_0^p}{r_+^p} = \frac{\tilde{\eta}_+^p \alpha^p(\omega_+)}{\tilde{\eta}_0^p} = \alpha^p(\omega_+) \frac{\omega_0 - 1}{\omega_+ - 1}, \tag{54}$$

where the second equality is obtained by substituting the values for $\tilde{\eta}_+$ and $\tilde{\eta}_0$ (recall that the hypocycloid is cusped when $\omega = \omega_0$ or $\omega = \omega_+$). From the fact that $h(\omega_0) = h(\omega_+) = 0$, we get that

$$\frac{\omega_0 - 1}{\omega_+ - 1} = \left(\frac{\omega_0}{\omega_+} \right)^p. \tag{55}$$

Thus, the relation (54) becomes

$$\left(\frac{r_0}{r_+} \right)^p = \left(\alpha(\omega_+) \frac{\omega_0}{\omega_+} \right)^p,$$

which gives

$$\frac{r_0}{r_+} = \alpha(\omega_+) \frac{\omega_0}{\omega_+}. \tag{56}$$

We see now that $\omega_+/\omega_0 > 1$ and that

$$f_{\omega_+,1}\left(\frac{\omega_+}{\omega_0}\right) = \left(\frac{\omega_+}{\omega_0}\right)^p (1-\omega_0) - (1-\omega_+).$$

From (55) we get

$$f_{\omega_+,1}\left(\frac{\omega_+}{\omega_0}\right) = (1-\omega_+) - (1-\omega_+) = 0.$$

Theorem 4.1(b) now assures us that

$$\omega_+/\omega_0 = \gamma(\omega_+) = \alpha(\omega_+),$$

and (56) immediately yields $r_0 = r_+$. An entirely analogous argument and the use of Theorem 4.2(b)(i) can be used to prove that $r_0 = r_-$ as well.

Having established that ω_0, ω_+, and (for p odd) ω_- yield the same convergence factor r_*^p, we now proceed to show that all the other values that can be chosen for the parameter ω give slower convergence. The proof will examine all feasible ranges of ω-values, determine the η-value that is required (for some given ω) for a convergence factor r_*^p, and then establish that, with this choice of η, the domain $U_\eta(\omega)$ does not include all the eigenvalues of $\tilde{\sigma}(J)$; stated in other words, SOR does not converge. A smaller η should be used to make SOR converge, and that is a restatement of the fact that the ω-values, in the range considered, cannot attain the convergence factor r_*^p. In order to proceed with the proof, we need the following technical lemmas. Their proof is in Appendix D.

LEMMA 6.1. *The root $\gamma(\omega)$ of $f_{\omega,1}(\lambda)$, as defined in Theorem* 4.1(b), *is an increasing function of ω. The same holds for the root $\gamma(\omega)$ as defined in Theorem* 4.2(b)(i).

LEMMA 6.2. *The functions $\gamma^p(\omega)/(\omega-1)$ and $\gamma(\omega)/\omega$ are strictly increasing for $\omega > p/(p-1)$ and strictly decreasing for $\omega < \omega^*$ with ω^* as defined by Theorem* 4.2(b). *The quantity $\gamma(\omega)$ is as in Theorem* 4.1(b) *in the first case and as in Theorem* 4.2(b)(i) *in the second.*

LEMMA 6.3. *The function $U_{p-1}(t)/U_{p-2}(t)$, where $U_{p-1}(t)$ and $U_{p-2}(t)$ are the Chebyshev polynomials of degrees $p-1$ and $p-2$ respectively, is a* *strictly increasing function in* $(\cos(\pi/p),1)$. *Therefore*

$$\frac{U_{p-1}(t)}{U_{p-2}(t)}<\frac{U_{p-1}(1)}{U_{p-2}(1)}=\frac{p}{p-1}.$$

We now consider the case $0<\omega<\omega_0<p/(p-1)$. Theorem 4.1 dictates that, in this range, $\alpha(\omega)=1$; hence, to attain a convergence factor equal to ${r_*}^p=r_0^p$, we need to use $\eta=\tilde{\eta}_0$. For $0<\omega<1$ the hypocycloid is rotated counterclockwise by an angle π/p, and it meets the real positive semiaxis at $d_{\min}$ as defined in (45), with parameters provided by (47). On the other hand, if $1<\omega<\omega_0$, the hypocycloid is not rotated. Its parameters are given by (44). Since $1/\tilde{\eta}_0^p=(p-1)(\omega_0-1)>(p-1)(\omega-1)$, the hypocycloid is shortened, and it cuts the real positive semiaxis at the point $d_{\min}$ as given by (46). For both ranges of ω the intersection point is

$$x(\omega)=\frac{1}{\omega\tilde{\eta}_0}+\frac{\omega-1}{\omega}\tilde{\eta}_0^{p-1}.$$

It is easy to verify that $dx(\omega)/d\omega=(\tilde{\eta}_0^{p+1}-1)/\omega_0^2\tilde{\eta}_0^2>0$. Since $\omega<\omega_0$, we get $x(\omega)<x(\omega_0)=\tilde{\rho}(J)$, which means that $\tilde{\sigma}(J)\not\subset U_{\tilde{\eta}_0}(\omega)$ $\forall\omega\in(0,\omega_0)$. Therefore, no ω-value in this range can attain the convergence factor r_*^p.

We now proceed to the case $\omega_0<\omega\leqslant p/(p-1)$. The hypocycloid is as in the case of $1<\omega<\omega_0$. Since, by Theorem 4.1, we have $\alpha(\omega)=1$, we still need an $\eta=\tilde{\eta}_0$. Now $1/\tilde{\eta}_0^p=(p-1)(\omega_0-1)<(p-1)(\omega-1)$, and the hypocycloid is stretched. We thus need to prove that the inner point of intersection with the real positive semiaxis (and *not* the point $d_{\max}$) is less than $\tilde{\rho}(J)$. This intersection point is

$$x=\frac{1}{\omega}\left(\frac{1}{\tilde{\eta}_0}t+(\omega-1)\tilde{\eta}_0^{p-1}T_{p-1}(t)\right),$$

where t is the (unique) solution of

$$U_{p-2}(t)=\frac{1}{(\omega-1)\tilde{\eta}_0^p}$$

in $(\cos(\pi/p),1)$. By combining the two equations and using an elementary property of Chebyshev polynomials, we get

$$x = \frac{1}{\omega \tilde{\eta}_0} \frac{U_{p-1}(t)}{U_{p-2}(t)}.$$

We now show that $x < \tilde{\rho}(J)$. Recall that, by construction,

$$\tilde{\rho}(J) = \frac{1}{\omega_0 \tilde{\eta}_0} + \frac{\omega_0 - 1}{\omega_0} \tilde{\eta}_0^{p-1}$$

and that $1/\tilde{\eta}_0^p = (p-1)(\omega_0 - 1)$. These relations yield

$$\tilde{\rho}(J) = \frac{p/(p-1)}{\omega_0 \tilde{\eta}_0}.$$

Thus, our claim $x < \tilde{\rho}(J)$ becomes equivalent to

$$\frac{U_{p-1}(t)}{U_{p-2}(t)} < \frac{p}{p-1} \frac{\omega}{\omega_0}.$$

By Lemma 6.3 and the fact that $\omega > \omega_0$, we see this last relation to be true. We conclude that, in this range as well, $\tilde{\sigma}(J) \not\subset U_{\tilde{\eta}_0}(\omega)$ and the convergence factor $r_*{}^p$ cannot be obtained.

We now investigate to the ranges of positive ω-values that have $\alpha(\omega) > 1$. The parameters of the hypocycloid are as in the previous case. Theorem 4.1(b) indicates that in order to obtain a convergence factor $r_*{}^p = r_+^p$ we need an $\eta = \tilde{\eta}_+ \alpha(\omega_+)/\alpha(\omega) = \tilde{\eta}_+ \gamma(\omega_+)/\gamma(\omega)$. We first show that this η-value results in a stretched hypocycloid. Indeed, the hypocycloid will be stretched iff

$$\eta^p > \frac{1}{(p-1)(\omega - 1)}$$

or, equivalently, iff

$$\frac{\gamma^p(\omega_+)}{\gamma^p(\omega)} > \frac{1}{(p-1)(\omega-1)} \frac{1}{\tilde{\eta}_+^p} = \frac{\omega_+ - 1}{\omega - 1},$$

where the last equality comes by recalling that, by construction of ω_+, $1/\tilde{\eta}_+^p = (p-1)(\omega_+ - 1)$. The inequality becomes

$$\frac{\gamma^p(\omega_+)}{\omega_+ - 1} > \frac{\gamma^p(\omega)}{\omega - 1}.$$

Since $\omega_+ > \omega > 0$, Lemma 6.2 assures us that this last inequality is true, and the hypocycloid stretched.

We now show that x, the inner point of intersection of this hypocycloid with the real positive semiaxis, is less than $\tilde{\rho}(J)$. As in the previous case, we find that

$$x = \frac{1}{\omega\eta}\frac{U_{p-1}(t)}{U_{p-2}(t)},$$

with t the unique solution of

$$U_{p-2}(t) = \frac{1}{(\omega-1)\eta^p}$$

in $(\cos(\pi/p), 1)$, and that

$$\tilde{\rho}(J) = \frac{p/(p-1)}{\omega_+\tilde{\eta}_+}.$$

Therefore $x < \tilde{\rho}(J)$ is equivalent to

$$\frac{U_{p-1}(t)}{U_{p-2}(t)} < \frac{p}{p-1}\frac{\omega}{\omega_+}\frac{\eta}{\tilde{\eta}_+} = \frac{p}{p-1}\frac{\gamma(\omega_+)/\omega_+}{\gamma(\omega)/\omega},$$

where the last equality comes from substituting the value of η. Since $\omega < \omega_+$, Lemmas 6.2 and 6.3 verify that the inequality is true, so that these ω-values cannot yield the convergence factor $r_*{}^p$.

The last range of positive ω-values to be examined is for $\omega > \omega_+$. We need, in this case too, $\eta = \tilde{\eta}_+\gamma(\omega_+)/\gamma(\omega)$. By application of Lemma 6.2 we determine that, this time,

$$\frac{\gamma^p(\omega_+)}{\omega_+ - 1} < \frac{\gamma^p(\omega)}{\omega - 1},$$

and by proceeding as in the previous case we see that this time the hypocycloid is *shortened*. Again, we prove that the (single) point of intersection of the hypocycloid with the real positive semiaxis $x = d_{\max}$ is less than $\tilde{\rho}(J)$. That is, we need to prove

$$x = \frac{1 + (\omega - 1)\eta^p}{\omega\eta} < \tilde{\rho}(J) = \frac{p/(p-1)}{\omega_+ \tilde{\eta}_+},$$

or, by substituting the value of η,

$$1 + (\omega - 1)\frac{\gamma^p(\omega_+)}{\gamma^p(\omega)}\tilde{\eta}_+^p < \frac{p}{p-1}\frac{\omega\gamma(\omega_+)}{\omega_+\gamma(\omega)}.$$

By recalling that $1/\tilde{\eta}_+^p = (p-1)(\omega_+ - 1)$, after a bit of algebra, we get

$$p - 1 + \frac{\omega - 1}{\omega_+ - 1}\frac{\gamma^p(\omega_+)}{\gamma^p(\omega)} < p\frac{\omega\gamma(\omega_+)}{\omega_+\gamma(\omega)}.$$

The fact that $f_{\omega,1}(\gamma(\omega)) = f_{\omega_+,1}(\gamma(\omega_+)) = 0$ yields

$$\frac{\omega - 1}{\omega_+ - 1} = \frac{\gamma^{p-1}(\omega)[\omega - \gamma(\omega)]}{\gamma^{p-1}(\omega_+)[\omega_+ - \gamma(\omega_+)]}.$$

Thus, the relation to be proved becomes

$$p - 1 + \frac{\omega - \gamma(\omega)}{\omega_+ - \gamma(\omega_+)}\frac{\gamma(\omega_+)}{\gamma(\omega)} < p\frac{\omega\gamma(\omega_+)}{\omega_+\gamma(\omega)}.$$

After some algebraic transformations, this becomes

$$(p-1)\omega_+^2\omega\left(\frac{\gamma(\omega)}{\omega} - \frac{\gamma(\omega_+)}{\omega_+}\right) < p\gamma(\omega_+)\omega\omega_+\left(\frac{\gamma(\omega)}{\omega} - \frac{\gamma(\omega_+)}{\omega_+}\right).$$

Since $\omega > \omega_+$, Lemma 6.2 guarantees that the quantity in large parentheses in the relation above is positive. The inequality simplifies to

$$\frac{p-1}{p}\omega_+ < \gamma(\omega_+).$$

Theorem 4.1(b) suggests that this inequality is true. We have now effectively proved that the ω-values in this range as well cannot yield the convergence factor $r_*{}^p$.

Let us now consider negative values of ω. The discussion in Section 4 revealed that the case of odd p is different from the case of even p. We first consider even values of p. The boundary of $U_\eta(\omega)$ is a hypocycloid rotated counterclockwise by an angle π/p. It cuts the real positive semiaxis at $x = d_{\min}$, which by (46) and (48) is seen to be

$$x = \frac{-1+(1-\omega)\eta^p}{\omega\eta}.$$

If we want to obtain the convergence factor $r_* = r_0$ we have to employ $\eta = \tilde{\eta}_0/\alpha(\omega)$. As usual, we will prove that, in this case as well, this η-value results in $\tilde{\sigma}(J) \not\subset U_\eta(\omega)$ and SOR does not converge. Indeed, for that value of η, either $d_{\min} < 0$ and $U_\eta(\omega)$ is empty (so that there is nothing more to prove), or, as we now prove, the point x is less than $\tilde{\rho}(J)$. The relation

$$x = \frac{-1+(1-\omega)\eta^p}{\omega\eta} = \alpha(\omega)\frac{-1+(1-\omega)\tilde{\eta}_0^p/\alpha^p(\omega)}{\omega\tilde{\eta}_0} < \tilde{\rho}(J) = \frac{p/(p-1)}{\omega_0\tilde{\eta}_0}$$

holds iff (recall that $\omega < 0$)

$$-\alpha(\omega) + (1-\omega)\frac{\tilde{\eta}_0^p}{\alpha^{p-1}(\omega)} > \frac{\omega}{\omega_0}\frac{p}{p-1},$$

and, by Theorem 4.2(a) (and because p is even)

$$\gamma(\omega) - \frac{1-\omega}{(\omega_0-1)(p-1)}\frac{1}{\gamma^{p-1}(\omega)} > \frac{\omega}{\omega_0}\frac{p}{p-1}.$$

After some algebra, and by using the fact that $f_{\omega,1}(\gamma(\omega))=0$, we get

$$\gamma^{p}(\omega)\left(\omega_0-\frac{p}{p-1}\right)<\frac{1-\omega}{\omega_0-1}\left(\frac{p}{p-1}-\omega_0\right).$$

Since $1<\omega_0<p/(p-1)$, we get

$$\gamma^{p}(\omega)>-\frac{1-\omega}{\omega_0-1}.$$

Since p is even and the quantity in the right-hand side is negative, we get the desired result.

We now proceed to the case of p odd. The case $\omega_-<\omega\leqslant\omega^*$, where ω^* is defined in Theorem 4.2, is treated in an entirely analogous way to the case $p/(p-1)<\omega<\omega_+$. Also, the case $\omega<\omega_-$ is treated like the case $\omega_+<\omega$. It remains to handle the case $\omega^*<\omega<0$. Again, if we want to obtain the convergence factor $r_*{}^p=r_-^p$, we need to introduce $\eta=\tilde{\eta}_-\alpha(\omega_-)/\alpha(\omega)=\tilde{\eta}_-\gamma(\omega_-)/\gamma(\omega)$, where $\gamma(\omega)$ is defined in Theorem 4.2(b)(iii). We now show that this choice for the parameter η yields a stretched hypocycloid for the boundary of $U_\eta(\omega)$. The hypocycloid is stretched iff

$$\eta^p=\frac{\alpha^p(\omega_-)}{\alpha^p(\omega)}\tilde{\eta}_-^p>\frac{1}{(p-1)(1-\omega)}$$

or, equivalently, iff

$$\frac{\alpha^p(\omega_-)}{(p-1)(1-\omega_-)}>\frac{\alpha^p(\omega)}{(p-1)(1-\omega)}. \tag{57}$$

Theorem 4.2(b)(i) implies that $\alpha^p(\omega_-)=-\gamma^p(\omega_-)$, and from the relation (33) we get

$$\frac{\alpha^p(\omega_-)}{(p-1)(1-\omega_-)}>1. \tag{58}$$

Also, Theorem 4.2(b)(iii) reveals that

$$\alpha^p(\omega)=(1-\omega)U_{p-2}(t^*)$$

with $t^* \in (\cos(\pi/p), 1)$; therefore,

$$\alpha^p(\omega) = (1-\omega)U_{p-2}(t^*) < (1-\omega)(p-1),$$

or

$$\frac{\alpha^p(\omega)}{(1-\omega)(p-1)} < 1. \tag{59}$$

The relations (58) and (59) reveal that (57) is valid, and the boundary of $U_\eta(\omega)$ stretched.

We are now going to show that the inner point of intersection x of this hypocycloid with the real positive semiaxis is less than $\tilde{\rho}(J)$—equivalently, that

$$x = \frac{1}{\omega\eta}\frac{U_{p-1}(t')}{U_{p-2}(t')} < \frac{p/(p-1)}{\omega_-\tilde{\eta}_-} = \tilde{\rho}(J), \tag{60}$$

where t' is the unique solution of

$$U_{p-2}(t') = \frac{1}{(1-\omega)\eta^p}$$

in $(\cos(\pi/p), 1)$. Notice that (58) implies $\alpha^p(\omega_-)\tilde{\eta}_-^p > 1$, so that we get

$$\eta^p = \frac{\alpha^p(\omega_-)}{\alpha^p(\omega)}\tilde{\eta}_-^p > \frac{1}{\alpha^p(\omega)},$$

which yields

$$\frac{1}{(1-\omega)\eta^p} < \frac{\alpha^p(\omega)}{1-\omega}.$$

The relation (35) in Theorem 4.2(b)(iii) now gives

$$U_{p-2}(t^*) = \frac{\alpha^p(\omega)}{1-\omega} > \frac{1}{(1-\omega)\eta^p} = U_{p-2}(t').$$

Since $U_{p-2}(t)$ is an increasing function in $(\cos(\pi/p), 1)$, the conclusion is that $t^* > t'$.

Returning to our original task, the relation (60) to be proved is equivalent to

$$\frac{U_{p-1}(t')}{U_{p-2}(t')} < \frac{p}{p-1}\,\frac{\omega\eta}{\omega_-\tilde{\eta}_-} = \frac{p}{p-1}\,\frac{\omega\gamma(\omega_-)}{\omega_-\gamma(\omega)}.$$

By Theorem 4.2(b)(i), $\gamma(\omega_-)/\omega_- > (p-1)/p$, so it suffices to prove

$$\frac{U_{p-1}(t')}{U_{p-2}(t')} < \frac{\omega}{\gamma(\omega)}.$$

By (35) we get

$$\left(\frac{\gamma(\omega)}{\omega}\right)^p = \frac{\omega-1}{\omega^p}U_{p-2}(t^*),$$

and by combining this with (34) we get

$$\left(\frac{\gamma(\omega)}{\omega}\right)^p = \left(\frac{U_{p-2}(t^*)}{U_{p-1}(t^*)}\right)^p.$$

Thus, the relation to be proved becomes

$$\frac{U_{p-1}(t')}{U_{p-2}(t')} < \frac{U_{p-1}(t^*)}{U_{p-2}(t^*)}.$$

Since $t' < t^*$, by using Lemma 6.3, we get the desired result.

We have effectively shown that no other value of ω besides ω_0, ω_+, and (for p odd) ω_- can yield an SOR convergence factor better than or equal to ${r_*}^p = r_0^p = r_+^p = r_-^p$. We can therefore state

THEOREM 6.1. *If the block SOR relaxations are employed to solve the system* (16) *that has an associated block Jacobi matrix J of the form* (7) *and* *such that* J^p *has only nonnegative eigenvalues, then, with* $\tilde{\rho}(J)$ *as defined by* (37):

(a) *If* $\tilde{\rho}(J) > 0$, *then let* ω_0, ω_+ *be the only positive roots of* $h(\omega)$ (*as defined in* (53)) *in*

$$\left(1, \frac{p}{p-1}\right) \quad \text{and} \quad \left(\frac{p}{p-1}\tilde{\rho}(H)^{-p/(p-1)}, \frac{p^{p/(p-1)}}{p-1}\tilde{\rho}(J)^{-p/(p-1)}\right)$$

respectively. If the SOR relaxations are employed with ω *equal to either of these two values, then the optimal SOR convergence factor* $r_*^p = (p-1)(\omega_0 - 1)$ *is obtained. In addition, if* p *is odd, let* ω_- *be the unique negative root of* $h(\omega)$ *in*

$$\left(-\frac{p^2}{p-1}\tilde{\rho}(J)^{-p/(p-1)}, -\frac{p}{p-1}\tilde{\rho}(J)^{-p/(p-1)}\right).$$

Then ω_-, *as well as* ω_0 *and* ω_+, *yields the optimal convergence factor* r_*^p. *SOR converges in the standard sense only for* $\omega = \omega_0$.

(b) *If* $\tilde{\rho}(J) = 0$, *then* $\omega = 1$ *is the only value that yields the optimal convergence factor* $r_*^p = 0$. *SOR converges in the standard sense.*

This theorem reveals that, judging just from the point of view of the asymptotic convergence rate, the newly introduced ω-ranges that yield convergence in the extended sense cannot improve the convergence rate. In fact, the "usual" optimal value ω_0 helps in avoiding the aggregation step presented in the next section. The numerical tests discussed in Section 8, however, reveal that the convergence factor of SOR is much more insensitive to small perturbations of ω around the values ω_+ and ω_- than around ω_0. We discuss that point further in that section.

7. COMPUTING THE CONSTANTS FOR THE SUBVECTORS OF THE SOLUTION

As we commented upon in previous sections, if SOR is employed with a value of the parameter ω that is outside the interval $(0, p/(p-1))$, then it converges to a vector that is *not* the stationary probability vector. The subvectors of the vector to which SOR converges are known, by Theorem 3.2, to be parallel to the corresponding subvectors of the stationary probability vector. The partition of the abovementioned vectors to subvectors is assumed to be conformal with the partition of J in (7).

Since the matrix Q is assumed to be already in p-cyclic normal form (13), the partition of the SOR vector into subvectors is known. Therefore, by normalizing each subvector separately, so that its elements sum to one, we get the vector

$$\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_p)^T,$$

while the stationary probability vector is

$$\boldsymbol{\pi} = \left(\tau_1 \boldsymbol{\nu}_1^T, \dots, \tau_p \boldsymbol{\nu}_p^T\right),$$

where τ_i, $i = 1, \dots, p$, are appropriate positive constants.[1] In fact, each $\boldsymbol{\nu}_i$, $i = 1, \dots, p$, is the probability distribution of the Markov chain being in a particular state, within the subset of states defined by the block Q_{ii} in (13), conditioned upon the fact that the Markov chain is in that subset. Then τ_i is the total probability of the Markov chain being in that subset of states.

The vector

$$\boldsymbol{\tau} = (\tau_1, \dots, \tau_p)$$

is easily seen, from the observations above, to be a probability vector. Moreover, it is the stationary probability vector of another Markov chain, that is,

$$\boldsymbol{\tau} G = \mathbf{0}, \tag{61}$$

where the infinitesimal generator of the new Markov chain is $G = (g_{ij})$ with $g_{ij} = \boldsymbol{\nu}_i^T Q_{ij} \mathbf{e}$, where $\mathbf{e}$ is a column vector of all ones, of appropriate size. This Markov chain examines transitions from a whole block of the original Q-matrix to another block, as if these blocks were single states. It is very often met with in aggregation-disaggregation techniques for solving *nearly completely decomposable* (NCD) Markov problems (see e.g. [4, 9]). The matrix G is irreducible if the original chain Q is itself irreducible. This is

[1]We transposed $\boldsymbol{\nu}_i$ in the vector $\boldsymbol{\pi}$ above, because we assumed that $\boldsymbol{\nu}_i$ are column vectors, derived from SOR, while $\boldsymbol{\pi}$ is a row vector.

easy to see because irreducibility implies that every state is reachable from any other, so that every block corresponding to a state in G is certainly reachable from every other block.

Since we have assumed irreducibility of Q, Equation (61) along with the normalizing condition $\boldsymbol{\tau}\mathbf{e} = 1$, uniquely determines the vector $\boldsymbol{\tau}$. The system (61) is of dimension p, so it is trivial to solve if p is not excessively big. The fact that Q has the special p-cyclic form (13) makes it even easier. We can, for example, assign an arbitrary starting value at τ_1, then perform the recursive scheme.

$$\tau_i = -\tau_{i-1}\frac{\boldsymbol{\nu}_{i-1}^T Q_{i-1,i}\mathbf{e}}{\boldsymbol{\nu}_i^T Q_{ii}\mathbf{e}}, \qquad i = 2,\ldots,p, \tag{62}$$

and finally normalize $\boldsymbol{\tau}$ so that its elements add up to one.

If the original chain is NCD, the quantities $\boldsymbol{\nu}_i^T Q_{ii}\mathbf{e}$ will be very small and numerical difficulties may arise in the application of the recursive scheme (62). If this is the case, however, some of the aggregation-dissaggregation techniques would be more appropriate, for solving the original problem (16), than applying block SOR to the whole state space as is examined here.

So it seems fitting to say that, for all domains of possible application, the forward recursive scheme (62) (or a corresponding backward scheme) is a fairly stable and efficient way (requires less operations than a single block SOR iteration step, for *any* p) to retrieve the stationary probability vector $\boldsymbol{\pi}$ from the vector $\boldsymbol{\nu}$ to which SOR converges, in the case where the parameter ω is chosen such that SOR does not converge to $\boldsymbol{\pi}^T$ directly.

8. AN EXAMPLE MODEL AND NUMERICAL TESTS

In this section, we present a class of continuous time Markov chains, with p-cyclic infinitesimal generators of the form (13), derived from the modeling of open queueing networks with blocking. We then discuss the numerical results from the application of SOR to matrices derived from this class of models.

Assume a open tandem of p queues. Customers arrive at the first queue according to a Poisson point arrival process, with rate μ_0. They then wait to be served by the server associated with the first queue, and when this is done they proceed to the next queue. They continue in this fashion, until they are served by the server associated with the last (pth) queue. They then depart from the system. The service times at all servers are exponentially distributed, with the ith server having a mean service time $1/\mu_i$, $i = 1, \dots, p$. All queues have finite capacities N_i, $i = 1, \dots, p$. A customer that arrives at the system and finds the first queue full is lost. A customer that is about to start service at the ith queue and finds the $(i+1)$th queue full waits for an empty space in the $(i+1)$th queue before starting its service. In the meantime, the server at the ith queue stays idle. This type of blocking is called *blocking before service* or *communication blocking*, as opposed to the *blocking after service* or *manufacturing blocking*, which would be more realistic in this context, but is slightly more involved in its state description. For details on blocking mechanisms and comparisons between the ones just referred to see [13, 1]. This model was used by Mitra and Tsoukas [10], as an illustrative example for their results on the effect of the ordering of the state space on the convergence of point Gauss-Seidel iteration.

The state space of the model in terms of a Markov chain description is

$$S = \left\{ \mathbf{n} = (n_1, \dots, n_p) \,\middle|\, 0 \leqslant n_i \leqslant N_i,\ i = 1, \dots, p \right\},$$

where n_i is the number of customers in the ith queue.

The transitions from a state $\mathbf{n}$ are

$$\begin{aligned} T_0(\mathbf{n}) &= T_0\big((n_1, \dots, n_p)\big) = (n_1 + 1, \dots, n_p), \\ T_i(\mathbf{n}) &= T_i\big((n_1, \dots, n_i, n_{i+1}, \dots, n_p)\big) \\ &= (n_1, \dots, n_i - 1, n_{i+1} + 1, \dots, n_p), \qquad i = 2, \dots, p-1, \\ T_p(\mathbf{n}) &= T_p\big((n_1, \dots, n_p)\big) = (n_1, \dots, n_p - 1). \end{aligned}$$

There are states for which some of these transitions are not possible. An ordering independent description of the infinitesimal generator Q can now be stated as

$$\begin{aligned} Q(\mathbf{n}, T_0(\mathbf{n})) &= \mu_0 I_{n_1 < N_1}, \\ Q(\mathbf{n}, T_i(\mathbf{n})) &= \mu_i I_{n_i > 0} I_{n_{i+1} < N_{i+1}}, \qquad i = 2, \dots, p, \\ Q(\mathbf{n}, T_p(\mathbf{n})) &= \mu_p I_{n_p > 0}, \end{aligned}$$

where $I_{\mathscr{F}}$ is the indicator function of the condition $\mathscr{F}$, and with all other nondiagonal elements being null. The diagonal elements are, of course,

$$Q(\mathbf{n},\mathbf{n}) = -\sum_{i=0}^{p} Q(\mathbf{n},T_i(\mathbf{n})).$$

Impose now the antiexicographical ordering on the state space. This causes the states to be ordered as

$$\begin{aligned}(0,0,\dots,0) &< (1,0,\dots,0) < (2,0,\dots,0) < \cdots < (N_1,0,\dots,0)\\ &< (0,1,\dots,0) < \cdots < (N_1,1,\dots,0)\\ &< (0,2,\dots,0) < \cdots < (N_1,N_2,\dots,N_p).\end{aligned}$$

The antiexicographical ordering may be obtained by defining

$$\begin{aligned}c_1 &= 1,\\ c_i &= \prod_{k=1}^{i-1}(N_k+1), \qquad i=2,\dots,p,\end{aligned}$$

and letting

$$\operatorname{order}(\mathbf{n}) = 1 + \sum_{k=1}^{p} c_k n_k.$$

If we now partition the state space S into the sets

$$S_1 = \{\mathbf{n}\in S \mid 1 \leqslant \operatorname{order}(\mathbf{n}) \leqslant N_1+1\} = \{\mathbf{n}\in S \mid n_j = 0,\ 1<j\leqslant p\},$$

$$\begin{aligned}S_i &= \left\{\mathbf{n}\in S \,\middle|\, \prod_{k=1}^{i-1}(N_k+1)+1 \leqslant \operatorname{order}(\mathbf{n}) \leqslant N_i\prod_{k=1}^{i-1}(N_k+1)\right\}\\ &= \{\mathbf{n}\in S \mid n_i \neq 0;\ n_j = 0,\ i<j\leqslant p\}, \qquad i=2,\dots,p,\end{aligned}$$

we obtain an infinitesimal generator of the block form

$$\begin{pmatrix} Q_{1,1} & Q_{1,2} & 0 & \cdots & 0 & 0 \\ 0 & Q_{2,2} & Q_{2,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{p-1,p-1} & Q_{p-1,p} \\ Q_{p,1} & Q_{p,2} & Q_{p,3} & \cdots & Q_{p-1,p} & Q_{p,p} \end{pmatrix}$$

where the states in the diagonal blocks Q_{ii} are exactly the states contained in the set S_i. Notice the factorial increase of the size of the blocks as the index increases.

In order to get a p-cyclic, infinitesimal generator, we impose the following modifications on the model:

(a) Any departure from the pth (last) queue is prohibited when there are customers in any intermediate queue.

Notice that this restriction is nullified if $p = 2$. To avoid deadlocks and maintain the irreducibility of Q, we insist on

(b) $N_p = \Sigma_{i=1}^{p-1} N_i$.

and

(c) No further arrivals to the system (1st queue) are permitted when the total population in the system, $\Sigma_{i=1}^{p-1} N_i$, reaches the limit N_p.

With this arrangement, in the worst case, all customers will be accumulated in the last queue and then depart without being blocked by customers in the intermediate queues. With these modifications the matrix Q becomes p-cyclic. The blocks Q_{ij}, $1 \leqslant i, j \leqslant p-1$, remain as before. The blocks Q_{pp} and Q_{p1}, however, change. Some states are not reachable [e.g. the "full" state $(N_1, \dots, N_p)$] and are removed from the system. The cardinality of the state space, though, remains of the same order of magnitude.

We hasten to point out that we do not propose block SOR as the best method for solving this model. While the first $p-1$ diagonal blocks are narrow banded and would have a "nice" LU factorization, the block Q_{pp} has elements all over the block, thus making an LU factorization expensive. Taking into account that this block has a size in the order of the entire matrix dimension, the LU factorization preprocessing before applying SOR would, at least for big matrices, be almost as expensive as choosing a direct method to solve the problem. In addition to that, we are not aware of any analytical method for determining $\bar{\rho}(J)$. The reason for choosing this model for

TABLE 1

CONFIGURATION OF FIRST EXAMPLE MODEL

Number of queues (p)	3
Arrival rate (μ_0)	$6a$
Service rate at 1st queue	a
Capacity of 1st queue	2
Service rate at 2nd queue	$2a$
Capacity of 2nd queue	2
Service rate at 3rd queue	$3a$
Capacity of 3rd queue	4

numerical tests, rather than using artificially chosen "nice" examples, is that it is derived from a physical model and illustrates how p-cyclic Markov chains may arise in stochastic modeling. We point out that it is not guaranteed that the eigenvalues of J^p are nonnegative. In what follows, we present particular configurations that satisfy this assumption.

We are now ready to consider the first test case. Consider the configuration shown in Table 1. The dimension of the matrix Q is 27. The spectrum of J^p, excluding the zero eigenvalues, is

$$\sigma(J^p) \setminus \{0\} = \{1, 0.0153\},$$

each eigenvalue having multiplicity 3. The theory-relevant quantities are summarized in Table 2. We applied the block SOR to this example, varying the parameter ω. The SOR control parameters are shown in Table 3. For

TABLE 2

RELEVANT QUANTITIES FOR FIRST EXAMPLE

$\bar{\rho}(J)$	0.2482
ω_0	1.0023
ω_+	20.4971
ω_-	-21.4994
ω^*	-3.0

TABLE 3

CONTROL PARAMETERS FOR SOR RUNS

Max iter. no.	Convergence threshold
100	1.E−8

each ω-value we performed the SOR relaxations twice, with a different normalizing and convergence method each time. In the first way, we normalized the whole vector, so that its elements summed to one. We then applied a relative error convergence test in the current and previous (normalized) iterates. In the second way, we normalized each subvector separately, so that its elements summed to one. We then applied the relative convergence test in each subvector (of the current and the previous iterate) separately. The biggest difference was taken as the measure to be compared against the tolerance. We then normalized the *whole* vectors and proceeded to the next iteration. The first way was expected to work when SOR was to converge to a single eigenvector, that is, for the ranges $(-\infty, \omega^*) \cup (0, p/(p-1)) \cup (p/(p-1), +\infty)$. It was expected to fail for the other values of ω, because in these cases SOR converges to a linear combination of eigenvectors, or eigenvectors and principal vectors, and convergence is achieved only in a subvector sense. The relative convergence tests were always performed using 1-norms, and the starting vector was always the same randomly selected probability vector. The results appear in Table 4. As one can see, the expectations were justified. Indeed, one has to introduce the subvector testing for $\omega \in [\omega^*, 0) \cup \{p/(p-1)\}$. The table also gives an idea of how the convergence factor behaves as a function of ω. We return to this point later in this section.

The next thing that was checked was if the vector to which SOR was converging really had subvectors parallel to the subvectors of the stationary probability vector. Remember that we expect to converge exactly to the solution for $\omega \in (0, p/(p-1))$, while for other values of ω we should get a vector that has subvectors parallel to the solution but is not itself parallel to it. We therefore measured the cosines of the angles between the subvectors of the SOR vector and the exact solution vector. We did the same for the whole vectors. Then we performed the aggregation step on the SOR vector and redid the computations. As a definition of the cosine between two vectors, the following quantity was used:

$$\cos(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2}.$$

The results appear in Table 5.

Notice how much the SOR vector differs from the exact solution when ω is out of the range $(0, p/(p-1))$, and how the aggregation procedure computes the solution from that vector. Notice also that the subvectors are always parallel to the subvectors of the solution. The multiplicative constants

TABLE 4
SOR RUNS FOR THE FIRST EXAMPLE

	Conv. test on whole vector		Conv. test on subvectors	
ω	Iterations	Difference	Iterations	Difference
−50	6	1.94822E−10	6	4.09178E−09
−42.8749	6	1.72011E−10	6	3.69649E−09
−35.7497	6	1.36572E−10	6	3.06018E−09
−28.6246	5	7.36076E−09	6	2.00641E−09
−21.4994	5	1.82815E−09	6	4.06507E−10
−16.8746	5	4.24944E−09	6	1.98749E−10
−12.2497	6	4.17391E−10	6	4.46819E−09
−7.62486	7	1.06168E−09	7	2.43425E−09
−3	100	5.2101E−05	28	8.87597E−09
−2	100	1.00044	48	6.49071E−09
−1	100	1.02482	66	9.72761E−09
0.25057	69	9.69756E−09	73	8.27675E−09
0.50114	31	7.89358E−09	32	8.56098E−09
0.75171	17	4.90079E−09	17	9.07058E−09
1	6	3.57141E−09	6	5.4176E−09
1.00228	6	1.81404E−10	6	2.68694E−10
1.12671	11	6.9472E−09	12	1.34947E−09
1.25114	17	4.21644E−09	17	6.38541E−09
1.37557	29	6.0551E−09	25	5.35477E−09
1.4	35	7.12828E−09	27	6.3883E−09
1.45	66	9.60623E−09	31	9.38502E−09
1.49	100	7.86724E−06	34	7.34211E−09
1.499	100	2.67062E−05	34	8.09736E−09
1.5	100	3.74922E−05	33	8.89229E−09
6.24929	7	2.47445E−09	7	9.92973E−09
10.9986	6	5.92999E−10	6	1.47006E−09
15.7479	5	5.22575E−09	6	8.59155E−11
20.4971	5	8.48517E−10	6	2.2343E−11
27.8729	5	3.89435E−09	6	1.06432E−09
35.2486	5	6.96793E−09	6	1.99407E−09
42.6243	5	9.07501E−09	6	2.65033E−09
50	6	1.483E−10	6	3.10806E−90

alternate in sign when ω is negative, something that is in accordance with Theorem 3.2.

We now return to the convergence factor, as a function of ω. In order to make more concrete the signs shown in Table 4, we plotted the actual convergence factor as a ratio of the relevant eigenvalues. For the ranges of ω where there were two eigenvalues of $\mathcal{L}_\omega$ having the dominant modulus, the

TABLE 5
AGGREGATION EFFECT ON THE SOR SOLUTION

	Before aggreg.	After aggreg.
	$\omega = -21.4994$	
Solution diff.	1.04978	1.46406E − 12
Subv. 1	1	1
Subv. 2	−1	1
Subv. 3	1	1
Whole	0.409337	1
	$\omega = -3$	
Solution diff.	1.02304	1.00438E − 09
Subv. 1	1	1
Subv. 2	−1	1
Subv. 3	1	1
Whole	0.68321	1
	$\omega = 1.00228$	
Solution diff.	1.05086E − 12	1.05091E − 12
Subv. 1	1	1
Subv. 2	1	1
Subv. 3	1	1
Whole	1	1
	$\omega = 1.5$	
Solution diff.	0.00775948	1.34558E − 09
Subv. 1	1	1
Subv. 2	1	1
Subv. 3	1	1
Whole	0.999974	1
	$\omega = 20.4971$	
Solution diff.	0.983533	1.9017E − 13
Subv. 1	1	1
Subv. 2	1	1
Subv. 3	1	1
Whole	0.423752	1

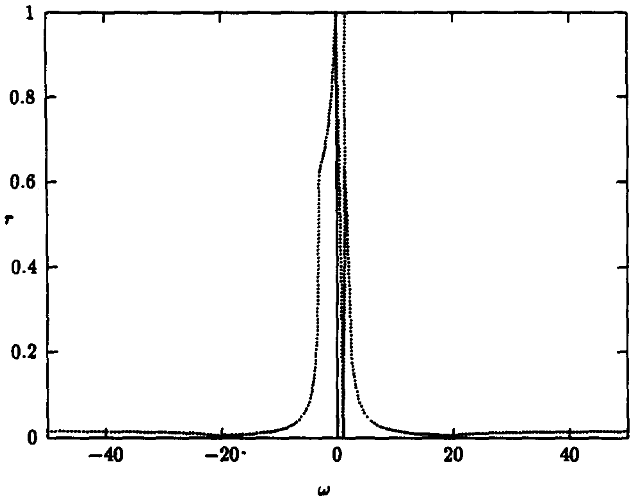

FIG. 7. Convergence factor as a function of ω: first example.

ratio of the third eigenvalue to the first was used. Otherwise, the ratio of the second eigenvalue to the first was computed. The plots appear in Figures 7 and 8. The second one scans a narrower range of ω, for clarity of presentation.

There are a number of things worthy of comment here. Notice first the discontinuity at $\omega = p/(p-1)$. This seems surprising at first, given the fact that the eigenvalues of $\mathscr{L}_\omega$ are continuous functions of ω. Remember, however, that in the interval $(0, p/(p-1))$ we are measuring the convergence ratio for convergence in the standard sense, that is, we expect SOR to converge directly to the solution, while in the interval $[p/(p-1), +\infty)$ we just look for convergence in a subvector sense. What actually happens here is simply that the subdominant eigenvalue of $\mathscr{L}_\omega$ for ω strictly less than but close to $p/(p-1)$ is (a pth power of) a root of $f_{\omega,1}(\lambda)$, and therefore the eigenvector associated with the subdominant eigenvalue has itself subvectors parallel to the solution. If we want to restrict ourselves to convergence in the subvector sense, we can ignore the subdominant eigenvalue, and compute (for this range) the new convergence factor as the ratio of the third eigenvalue to the dominant. Then the discontinuity disappears. This behavior is reflected very clearly in the data of Table 4. Notice that the numbers of iterations required using the two difference convergence criteria are very close to each other except for the range we are discussing. In this range, the first convergence criterion, which tries to extract the solution directly,

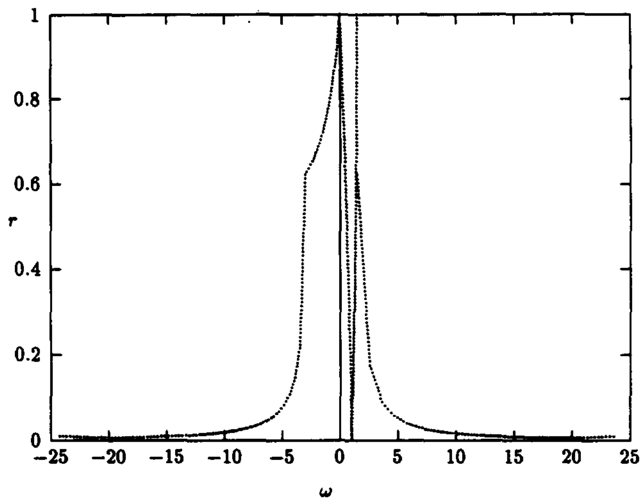

FIG. 8. More detailed plot of convergence factor for first example.

requires many more iterations than the second criterion, which tests just for subvector convergence and is satisfied much more quickly. Of course, if this second convergence test is used, the aggregation procedure should be employed to compute the exact solution. Thus, in this range, the aggregation procedure can be viewed as a way to save a significant number of iterations. This is not of significant practical use, though, because this range of ω-values, with either notion of convergence, gives a convergence factor far from the optimal.

The other thing that is worthy of comment, apart from the fact that the minimum points in the plot agree with the theoretically predicted values $\omega_0, \omega_+, \omega_-$, is how the convergence factor varies as a function of ω around the optimal ω-values. It is immediate that the convergence factor is much more sensitive around the "usual" optimum ω_0 (with any notion of convergence used) than it is around the points ω_+ and ω_-. Furthermore, as the absolute value of ω increases, the convergence factor remains practically invariant and suboptimally small. This behavior is of particular value in the cases where the quantity $\tilde{\rho}(J)$ is not known in advance and the optimal-values cannot be computed in advance. Then, a crude approximation of the parameter $\omega_+ \omega_-$ can be used, and as SOR proceeds, an adaptive procedure may be used to fine-tune ω. It is guaranteed that the convergence factor, during this adaptive procedure, will not fall excessively far from the optimal. This is not true if the optimum ω_0 is used. There small variations of ω may

TABLE 6
CONFIGURATION OF SECOND EXAMPLE MODEL

Number of queues (p)	4
Arrival rate (μ_0)	$5a$
Service rate at 1st queue	a
Capacity of 1st queue	2
Service rate at 2nd queue	$2a$
Capacity of 2nd queue	1
Service rate at 3rd queue	$3a$
Capacity of 3rd queue	2
Service rate at 4th queue	$4a$
Capacity of 4th queue	5

cause SOR to converge very slowly, thus making very difficult an adaptive procedure for the fine tuning of ω.

It should be noted that care has to be exercised when big ω-values are used. Theorem 3.2 shows that SOR converges to a vector with subvectors multiplied by factors that are powers of a quantity $[\alpha(\omega)]$ on the order of ω. If ω is excessively big, normalization of the successive iterates will cause the first subvectors (which are multiplied by smaller constants) to underflow. This phenomenon becomes particularly intense in the case of big p and small subblocks Q. The small subblocks may make $\bar{\rho}(J)$ small, and then (see Theorem 6.1) ω_+ and ω_- will be big. Thus the last subvectors of the eigenvector to which SOR will converge will be multiplied by very big constants. Fortunately, an alternative strategy is possible in this case. Since the subblocks of Q are small, and since J^p is a block diagonal matrix with subblocks having sizes corresponding to the diagonal blocks of Q, one can explicitly form the smallest subblock of J^p and compute its whole spectrum.[2] Then the quantity $\bar{\rho}(J)$ may be explicitly computed and an accurate estimation of ω_0 obtained. Then ω_0 can be used for the parameter ω without underflow problems.

We close this section by presenting the results from another numerical test, this time performed on a model that yields even p. The configuration of the model appears in Table 6. The dimension of the matrix Q is 63. The

[2] It is well known that the spectra of the diagonal blocks of J^p, excluding the zero eigenvalues, are identical, because all the subblocks J^p are permuted products of the blocks of J. The multiplicities of the eigenvalues may be different, though.

TABLE 7

RELEVANT QUANTITIES FOR SECOND EXAMPLE

$\bar{\rho}(J)$	0.3434
ω_0	1.0015
ω_+	8.4381

TABLE 8

SOR RUNS FOR THE SECOND EXAMPLE

	Conv. test on whole vector		Conv. test on subvectors	
ω	Iterations	Difference	Iterations	Difference
−40	6	1.63025E−10	7	1.7899E−10
−35	6	1.64729E−10	7	1.80171E−10
−30	6	1.67406E−10	7	1.82171E−10
−25	6	1.72192E−10	7	1.86074E−10
−20	6	1.82509E−10	7	1.95266E−10
−15	6	2.12222E−10	7	2.23862E−10
−10	6	3.58833E−10	7	3.73407E−10
−8.33333	6	5.49067E−10	7	5.73354E−10
−6.66667	6	1.0737E−09	7	1.14299E−09
−5	7	1.39449E−10	7	1.52565E−09
−3.33333	8	7.556E−09	9	3.26362E−09
−1.66667	15	5.62253E−09	16	5.18075E−09
0.250369	82	8.79351E−09	85	8.40534E−09
0.500738	36	7.52146E−09	37	8.47316E−09
0.751107	19	6.08289E−09	20	4.52853E−09
1	6	4.50229E−09	7	1.68885E−10
1.00148	6	2.66594E−10	6	7.42816E−10
1.08444	11	2.22376E−09	11	4.16564E−09
1.1674	21	1.9505E−09	21	2.8539E−09
1.2	21	3.9426E−09	18	5.45509E−09
1.25037	31	9.1348E−09	21	7.03837E−09
1.3	70	8.92455E−09	23	9.11008E−09
1.33	100	2.32291E−05	24	5.75635E−09
1.33333	100	3.17852E−05	24	5.77718E−09
3.10951	8	2.40896E−09	9	1.22264E−10
4.88569	6	4.21826E−09	7	5.7526E−10
6.66187	6	6.57421E−11	6	3.05447E−09
8.43805	5	2.04781E−09	6	1.14183E−09
16.3285	5	6.55537E−09	6	6.37151E−09
24.219	5	9.15467E−09	6	9.46943E−09
32.1095	6	1.38163E−10	7	1.46522E−10
40	6	1.43924E−10	7	1.54703E−10

TABLE 9
AGGREGATION EFFECT ON THE SOR SOLUTION

	Before aggreg.	After aggreg.
	$\omega = -5$	
Solution diff.	1.30447	2.55507E − 11
Subv. 1	−1	1
Subv. 2	1	1
Subv. 3	−1	1
Subv. 4	1	1
Whole	0.275772	1
	$\omega = 1.00148$	
Solution diff.	1.42831E − 12	1.4284E − 12
Subv. 1	1	1
Subv. 2	1	1
Subv. 3	1	1
Subv. 4	1	1
Whole	1	1
	$\omega = 1.33333$	
Solution diff.	0.0135222	9.08432E − 10
Subv. 1	1	1
Subv. 2	1	1
Subv. 3	1	1
Subv. 4	1	1
Whole	0.999944	1
	$\omega = 8.43805$	
Solution diff.	1.13225	2.20876E − 12
Subv. 1	1	1
Subv. 2	1	1
Subv. 3	1	1
Subv. 4	1	1
Whole	0.330797	1

spectrum of J^p, excluding the zero eigenvalues, is

$$\sigma(J^p) \setminus \{0\} = \{1, 0.0139\},$$

with all eigenvalues having multiplicity 4. The relevant quantities for this model are summarized in Table 7. The SOR relaxations were again performed under the same conditions with the previous example. The results of the runs appear in Table 8, and the effect of aggregation is shown in Table 9.

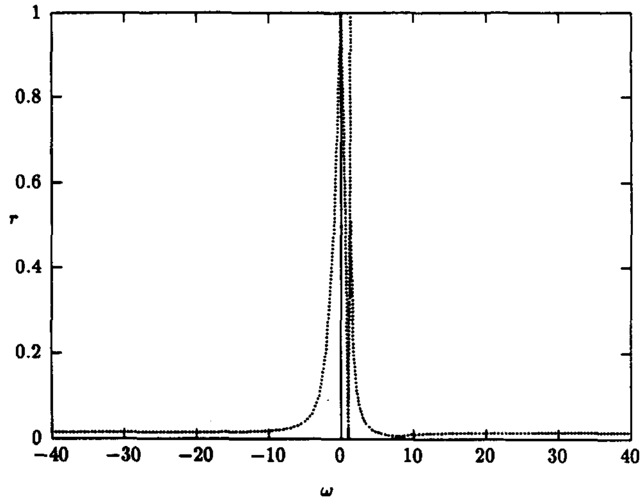

FIG. 9. Convergence factor as a function of ω: second example.

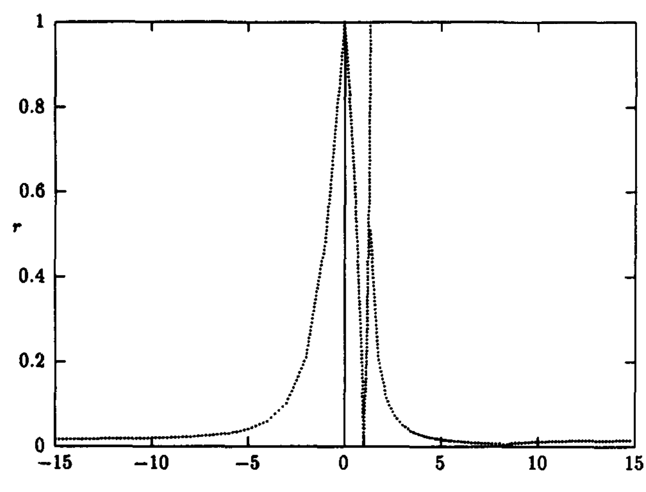

FIG. 10. More detailed plot of convergence factor for second example.

All comments apply to these results as well. In addition, notice that the behavior for negative ω is more uniform, as the theory predicts. There is no negative optimal ω-value (again consistent with the theory) but, as Figures 9 and 10 show, as ω becomes very negative, the convergence factor remains suboptimally small.

9. SUMMARY

In this work, we have studied the application of the block SOR method for computing the stationary probability distribution of Markov chains that possess p-cyclic infinitesimal generators. As became apparent, such Markov chains have a lot of structure that is reflected in the close connection of the eigensystem of the block Jacobi matrix J to the eigensystem of the SOR iteration matrix $\mathscr{L}_\omega$. This connection permitted an alternative statement of the usual conditions for semiconvergence in the form of conditions B1–B3 in Section 3 and led to the statement of the milder conditions C1–C2, under which convergence in a subvector sense is obtained. This new notion of convergence can be viewed as a generalization of the subvector convergence in the work of Courtois and Semal [5]. They introduced the subvector convergence for cases where the splitting associated with an iterative method is regular (and the spectral radius of the iteration matrix is 1), namely, for the Jacobi and Gauss-Seidel splittings, while we extended this notion to the SOR splitting, which is nonregular for $\omega > 1$ or $\omega < 0$. In addition, it became apparent that, under this generalized convergence, the parameter ω can, in principle, take any real value (apart from the null value 0).

We developed mathematical tools that are of value in investigating for the ω-ranges that yield convergence of SOR and finding the optimal relaxation factor values. We then applied these tools in the case when J^p has only nonnegative eigenvalues, determining the exact convergence intervals for convergence, in the standard and extended senses, and the optimal relaxation parameters. One can view the methods used in Sections 5 and 6 as extension of the methods used by Wild and Niethammer [20] and Eiermann, Niethammer, and Ruttan [6] for the singular homogeneous case. We note that the part of our results which is relevant to convergence in the standard sense for the case of J^p having nonnegative eigenvalues is in complete agreement with the results reported by Hadjidimos [8], so that this part of our work can be considered as an alternative proof of those results.

Although we did not treat any cases besides that of J^p having real nonnegative eigenvalues, it is obvious that the methods described in this paper can be directly used to deal with cases where the eigenvalues of J^p are (apart from the unit eigenvalue) nonpositive reals (see [20]) or just reals in general (see [6]). In fact this last case is of much more interest in the Markov chain setting, since most of the time Markov chains yield asymmetric matrices having nonregular spectra. Notice that if J^p contains nonnegative eigenvalues λ, $0 \leqslant \lambda \leqslant a^p$, nonpositive eigenvalues μ, $-b^p \leqslant \mu \leqslant 0$, and complex eigenvalues h, $|h| < \min(a^p, b^p)$, then this case is identical, from the point of view of convergence and the optimal ω, to the case of J^p having the same real eigenvalues only. That is because if the real eigenvalues are contained in $U_\eta(\omega)$, then the complex ones will be contained as well. The main problem is in identifying a pattern like this in the spectrum of J^p.

Another important factor is the determination of the quantity $\tilde{\rho}(J)$. Usually there is no means of determining it analytically, because of the asymmetric structure of the matrix Q. If this parameter is not known with adequate precision, the optimal ω-values cannot be determined accurately. This is why we consider as important the observation that variations around the newly introduced optimal ω-values ω_+ and (for odd p) ω_- do not affect the convergence factor significantly.

We mention that, since the optimal ω (for convergence in the usual sense), ω_0, is given by the same expression as for the nonsingular case, simply replacing the spectral radius $\rho(J)$ by $\tilde{\rho}(J)$, the results of Pierce, Hadjidimos, and Plemmons [14] relevant to the optimality of 2-cyclic SOR over any value of the parameter p, for the case of the eigenvalues of J^p having the same sign, carry over immediately to the setting discussed here as well. It is to be noted, though, that repartitioning the matrix Q in bigger subblocks, in order to get a 2-cyclic infinitesimal generator, may incur problems in the LU factorization preconditioning of the new big diagonal blocks that is required before the block SOR relaxations can start, and thus may not be practical in some cases.

Finally we point out that Galanis and Hadjidimos [22] have shown how to repartition p-cyclic matrices for optimum SOR convergence. Their methods should prove useful in our applications to Markov chains.

APPENDIX A. PROOF OF THEOREM 4.1

As we have already commented upon, 1 is a root of $f_{\omega,1}(\lambda)$ $\forall\omega$. We therefore dispense first with any negative roots of $f_{\omega,1}(\lambda)$ by showing that either they do not exist, or they have modulus less than 1. The derivative of $f_{\omega,1}(\lambda)$ is

$$f'_{\omega,1}(\lambda) = p\lambda^{p-2}\left(\lambda - \frac{p-1}{p}\omega\right). \tag{63}$$

For $\lambda < 0$ this derivative is negative if p is even and positive if p is odd. In the case of $0 < \omega < 1$ we have that

$$f_{\omega,1}(0) = \omega - 1 < 0.$$

If p is odd we have

$$f_{\omega,1}(\lambda) < f_{\omega,1}(0) < 0 \qquad \forall \lambda < 0,$$

and there cannot be a negative zero of $f_{\omega,1}(\lambda)$. If, on the other hand, p is even, then since $\lim_{\lambda \to -\infty} f_{\omega,1}(\lambda) = +\infty$, we do have a single negative root. Since

$$f_{\omega,1}(-1) = (-1)^p - \omega(-1)^{p-1} - (1-\omega) = 2\omega > 0,$$

the root is contained in the interval $(-1,0)$ and has modulus less than one. For the case $\omega > 1$, a very similar argument will show that for p even there is no negative root of $f_{\omega,1}(\lambda)$, while for p odd there is a negative root with modulus less than 1. We can therefore restrict ourselves in the study of $f_{\omega,1}(\lambda)$ to positive λ-values.

From equation (63) one sees that $f_{\omega,1}(\lambda)$ has a minimum at $[(p-1)/p]\omega$. In the case of $\omega \leqslant p/(p-1)$ this observation immediately leads us to the fact that there cannot be a root of $f_{\omega,1}(\lambda)$ greater than one [because, by (63), $f_{\omega,1}(\lambda) > f_{\omega,1}(1) = 0$ $\forall \lambda > 1$]. Furthermore, in the case of $\omega = p/(p-1)$, we get that $f_{\omega,1}(1) = f'_{\omega,1}(1) = 0$, so that the root 1 is multiple. It is easy to see that $f'_{\omega,1}(1) \neq 1$; therefore, 1 is a double root of $f_{\omega,1}(\lambda)$. We have thus justified the claims of part (a).

When $\omega > p/(p-1)$, the minimum occurs at a point greater than 1. This observation leads to

$$f_{\omega,1}\left(\frac{p-1}{p}\omega\right) < f_{\omega,1}(1) = 0, \tag{64}$$

which, after some elementary algebraic manipulation, yields the claimed relation (31). Furthermore, the fact

$$f_{\omega,1}(\omega) = \omega - 1 > 0,$$

in conjunction with (64), yields that there is a root $\gamma(\omega)$ of $f_{\omega,1}(\lambda)$, greater than 1, in $([(p-1)/p]\omega, \omega)$. Since $f_{\omega,1}(\lambda)$ is increasing for all $\lambda > [(p-1)/p]\omega$, there is no other real root greater than $\gamma(\omega)$.

To complete the proof, it remains to show that there are no complex roots that can affect the results discovered up to now. We will prove that, in all cases, all complex roots of $f_{\omega,1}(\lambda)$ have modulus strictly less than $\alpha(\omega)$. For the case of $0 < \omega < 1$, note that, for any complex root λ' of $f_{\omega,1}(\lambda)$, we must have

$$\lambda'^p = \omega\lambda'^{p-1} + 1 - \omega;$$

hence,

$$|\lambda'|^p \leqslant |\omega|\,|\lambda'|^{p-1} + |1-\omega|$$

and, since $\omega, 1-\omega > 0$,

$$|\lambda'|^p - \omega|\lambda'|^{p-1} - (1-\omega) \leqslant 0,$$

or

$$f_{\omega,1}(|\lambda'|) \leqslant 0.$$

From the discussion for the real roots we have seen that this may happen only if $|\lambda'| \leqslant 1$, so that there is no complex root with modulus exceeding $\alpha(\omega)$.

For the case of $\omega > 1$, for any complex root λ' of $f_{\omega,1}(\lambda)$ we must have

$$\frac{f_{\omega,1}(\lambda')}{\lambda' - 1} = 0,$$

or

$$\frac{\lambda'^p - 1}{\lambda' - 1} = \omega\frac{\lambda'^{p-1} - 1}{\lambda' - 1},$$

or

$$\sum_{k=0}^{p-1} \lambda'^k = \omega \sum_{k=0}^{p-2} \lambda'^k,$$

or

$$\lambda'^{p-1} = (\omega - 1) \sum_{k=0}^{p-2} \lambda'^{k};$$

hence,

$$|\lambda'|^{p-1} \leqslant |\omega - 1| \sum_{k=0}^{p-2} |\lambda'|^{k}.$$

We assume that $|\lambda'| > 1$; otherwise there is nothing to prove. By recalling that $\omega > 1$ we get

$$|\lambda'|^{p-1} \leqslant (\omega - 1) \frac{|\lambda'|^{p-1} - 1}{|\lambda'| - 1},$$

and, after some algebraic manipulation,

$$f_{\omega,1}(|\lambda'|) \leqslant 0.$$

This last relation immediately yields that $|\lambda'|$ cannot be greater than 1 when $1 < \omega < p/(p-1)$ (a contradiction to our hypothesis) and that it must be less than or equal to $\gamma(\omega)$ when $\omega > p/(p-1)$. In all cases we conclude that $|\lambda'| \leqslant \alpha(\omega)$.

The last thing that remains to be shown is that no complex root of $f_{\omega,1}(\lambda)$ achieves the modulus $\alpha(\omega)$. Assume there is some complex root λ such that $|\lambda| = \alpha(\omega)$. Then we can write $\lambda = \alpha(\omega)e^{j\theta}$ for some $\theta \neq k\pi$, $k \in \mathcal{Z}$, because λ is not real. Then, since $\alpha(\omega) = \gamma(\omega)$, from $f_{\omega,1}(\alpha(\omega)) = 0$ and $f_{\omega,1}(\lambda) = 0$ we get

$$\alpha(\omega)^{p}(1 - e^{jp\theta}) = \omega\alpha(\omega)^{p-1}(1 - e^{j(p-1)\theta}),$$

which, after some algebraic manipulation, yields

$$\alpha(\omega) \sin \frac{p\theta}{2} = \omega \sin \frac{(p-1)\theta}{2} e^{-j\theta/2}.$$

Since everything else is real, and since we cannot have $\sin(p\theta/2) =$

$\sin[(p-1)\theta/2] = 0$ (because $\theta \neq k\pi$), we must have $-\theta/2 = k\pi$, $k \in \mathcal{Z}$, or $\theta = 2k'\pi$, $k' \in \mathcal{Z}$. But this is a contradiction. The proof is now complete.

APPENDIX B. PROOF OF THEOREM 4.2

By (63) in Appendix A we see that, in all cases, $f'_{\omega,1}(\lambda) > 0$ $\forall \lambda > 0$. Since $f_{\omega,1}(1) = 0$, the only positive root is 1. We therefore restrict our attention to negative roots. If p is even, then, again by (63), we see that $f_{\omega,1}(\lambda)$ has a single local minimum on the negative semiaxis, at $[(p-1)/p]\omega$. Since $f_{\omega,1}(0) = \omega - 1 < 0$, the value the minimum is negative. Furthermore, since $\lim_{\lambda \to -\infty} f_{\omega,1}(\lambda) = +\infty$, there must be (only) one negative root. Indeed, one easily verifies that

$$f_{\omega,1}\left(-(1-\omega)^{1/p}\right) = \omega(1-\omega)^{(p-1)/p} < 0$$

and

$$f_{\omega,1}(\omega - 1) = (\omega - 1)\left[1 - (\omega - 1)^{p-2}\right] \geqslant 0,$$

so that there is a negative root of modulus greater than 1 in the interval $[\omega - 1, -(1-\omega)^{1/p})$ (actually, the root is exactly $\omega - 1$ iff $p = 2$).

To complete the proof of part (a) we need to show that all complex roots of $f_{\omega,1}(\lambda)$ have modulus strictly less than $\alpha(\omega)$. Let, therefore, a complex root be λ'. Then we have

$$\lambda'^p = \omega\lambda'^{p-1} + 1 - \omega,$$

or, by taking absolute values,

$$|\lambda'|^p \leqslant |\omega|\,|\lambda'|^{p-1} + |1-\omega|,$$

and, since $\omega < 0$ and $1 - \omega > 0$,

$$|\lambda'|^p + \omega|\lambda'|^{p-1} - (1-\omega) \leqslant 0.$$

By recalling that p is even, this becomes

$$(-|\lambda'|)^p - \omega(-|\lambda'|)^{p-1} - (1-\omega) \leqslant 0,$$

or

$$f_{\omega,1}(-|\lambda'|) \leqslant 0.$$

Since $-|\lambda'|$ is negative, the previous discussion implies that $-|\lambda'| \geqslant \gamma(\omega)$ or, equivalently, $|\lambda'| \leqslant \alpha(\omega)$. The fact that the inequality is sharp is established as in the proof of Theorem 4.1 (see Appendix A).

Let us now turn to part (b). Since p is odd, Equation (63) yields that $f_{\omega,1}(\lambda)$ has a maximum at $[(p-1)/p]\omega$. Observe that $f_{\omega,1}(0) = \omega - 1 < 0$ and that $\lim_{\lambda \to -\infty} f_{\omega,1}(\lambda) = -\infty$. This maximum has, therefore, to be greater or equal to zero if $f_{\omega,1}(\lambda)$ is to have negative real roots. The value of this maximum is

$$f_{\omega,1}\left(\frac{p-1}{p}\omega\right) = -\frac{1}{p-1}\left[\left(\frac{p-1}{p}\omega\right)^p - (p-1)(\omega-1)\right].$$

Therefore,

$$f_{\omega,1}\left(\frac{p-1}{p}\omega\right) \geqslant 0$$

iff

$$g(\omega) \equiv \left(\frac{p-1}{p}\omega\right)^p - (p-1)(\omega-1) \leqslant 0.$$

It is easy to see that the function $g(x)$ has a unique maximum on the negative semiaxis at $-p/(p-1)$. The corresponding maximum value is

$$g\left(-\frac{p}{p-1}\right) = 2(p-1) > 0.$$

This dictates that $g(\omega) \leqslant 0$ for $\omega \leqslant \omega_1 < -p/(p-1)$ or for $\omega > \omega_2 > -p/(p-1)$, where ω_1, ω_2 are roots of $g(x)$. Since $g(0) = p - 1 > 0$ and we are examining only negative values of ω, we conclude that $f_{\omega,1}(\lambda)$ has a negative root iff $\omega < \omega_1 \equiv \omega^* < -p/(p-1)$, where ω^* is the unique negative root of $g(x)$ in (32). It is easy to see that $g(-p) \leqslant 0$, and since we have already seen that

$$g\left(-\frac{p}{p-1}\right) > 0,$$

ω^* is contained in the interval claimed.

Also, if $\omega = \omega^*$, then since

$$g(\omega^*) = f_{\omega^*,1}\left(\frac{p-1}{p}\omega^*\right) = 0,$$

the root $\gamma(\omega^*)$ is equal to $[(p-1)/p]\omega^*$, and since it coincides with the point of the maximum, the root is multiple. It is easy to verify that

$$f''_{\omega^*,1}\left(\omega^* \frac{p-1}{p}\right) \neq 0,$$

so the root $\omega^* p/(p-1)$ is exactly double. In the case $\omega < \omega^*$ we get

$$\gamma(\omega) < \frac{p-1}{p}\omega,$$

and since

$$f_{\omega,1}(\omega) = \omega - 1 < 0,$$

we conclude that

$$\gamma(\omega) \in \left(\omega, \frac{p-1}{p}\omega\right),$$

as claimed. This root is simple, because $f'_{\omega,1}(\gamma(\omega)) > 0$. Furthermore, since in this range of ω-values $g(\omega)$ is strictly negative, we determine that (33) holds.

To complete the proof for subparts (i) and (ii) of part (b), we need to show that there are no complex roots of modulus greater or equal to the quantity $-\gamma(\omega) = \alpha(\omega)$. Assume that there is such a complex root

λ' of $f_{\omega,1}(\lambda)$ with modulus $|\lambda'| > -\gamma(\omega) = \alpha(\omega)$. From $f_{\omega,1}(\lambda') = 0$ and $f_{\omega,1}(\gamma(\omega)) = 0$ we get

$$\lambda'^p - \gamma^p(\omega) = \omega\left[\lambda'^{p-1} - \gamma^{p-1}(\omega)\right],$$

or

$$\frac{\lambda'^p - \gamma^p(\omega)}{\lambda' - \gamma(\omega)} = \omega \frac{\lambda'^{p-1} - \gamma^{p-1}(\omega)}{\lambda' - \gamma(\omega)},$$

or

$$\sum_{k=0}^{p-1} \lambda'^{p-1-k}\gamma^k(\omega) = \omega \sum_{k=0}^{p-2} \lambda'^{p-2-k}\gamma^k(\omega),$$

which gives

$$\lambda'^{p-1} + \gamma(\omega) \sum_{k=0}^{p-2} \lambda'^{p-2-k}\gamma^k(\omega) = \omega \sum_{k=0}^{p-2} \lambda'^{p-2-k}\gamma^k(\omega),$$

or

$$\lambda'^{p-1} = (\omega - \gamma(\omega)) \sum_{k=0}^{p-2} \lambda'^{p-2-k}\gamma^k(\omega).$$

By taking absolute values,

$$|\lambda'|^{p-1} \leqslant |\omega - \gamma(\omega)| \sum_{k=0}^{p-2} |\lambda'|^{p-2-k} |\gamma(\omega)|^k$$

$$= |\omega - \gamma(\omega)| \frac{|\lambda'|^{p-1} - |\gamma(\omega)|^{p-1}}{|\lambda'| - |\gamma(\omega)|}.$$

Since we assumed that $|\lambda'| > |\gamma(\omega)|$, we get

$$|\lambda'|^p - |\gamma(\omega)||\lambda'|^{p-1} \leqslant |\omega - \gamma(\omega)|\left(|\lambda'|^{p-1} - |\gamma(\omega)|^{p-1}\right).$$

Recall that $|\gamma(\omega)| = -\gamma(\omega)$, p is odd, and $\gamma(\omega) > \omega$. Using these facts, after some algebraic manipulation we get

$$|\lambda'|^p + \omega|\lambda'|^{p-1} \leqslant -\gamma^p(\omega) + \omega\gamma^{p-1}(\omega).$$

Since $f_{\omega,1}(\gamma(\omega)) = 0$, this becomes

$$|\lambda'|^p + \omega|\lambda'|^{p-1} \leqslant \omega - 1,$$

which leads to

$$f_{\omega,1}(-|\lambda'|) = (-|\lambda'|)^p - \omega(-|\lambda'|)^{p-1} - (1-\omega) \geqslant 0.$$

Since $-|\lambda'|$ is negative, from the discussion for the real roots we know that we must have $-|\lambda'| \geqslant \gamma(\omega)$, which contradicts our hypothesis. We therefore have proved that any complex root has modulus less or equal to $\alpha(\omega)$. The fact that the inequality is sharp is established as in Theorem 4.1.

Having proven subparts (i) and (ii) of part (b), we proceed to subpart (iii). From the previous discussion it is clear that when $\omega > \omega^*$, $g(\omega)$ is strictly positive. This fact immediately yields (36). The same discussion, however, revealed that there are no negative roots of $f_{\omega,1}(\lambda)$ in this case. Since there is only one positive root, namely the number 1, we know that there must be $(p-1)/2$ pairs of complex conjugate roots. We have to investigate these in order to determine $\alpha(\omega)$.

We let a root λ be represented as $\lambda = xe^{j\theta}$, with $x \in \mathfrak{R}$, $\theta \in (0,\pi)$. Notice that this is not the usual polar notation, although any complex number has a unique representation of this form. Using this notation, since λ is a root of $f_{\omega,1}(\lambda)$, by taking real and imaginary parts separately, we get

$$x^p \cos p\theta - \omega x^{p-1} \cos(p-1)\theta - (1-\omega) = 0$$

and

$$x^p \sin p\theta = \omega x^{p-1} \sin(p-1)\theta.$$

Furthermore, since $\theta \in (0,\pi)$ and $x \neq 0$ [because 0 is not a root of $f_{\omega,1}(\lambda)$], if we set $t = \cos\theta$ and divide the second of the equations above by $\sin\theta$, the previous pair of equations becomes

$$x^p T_p(t) - \omega x^{p-1} T_{p-1}(t) - (1-\omega) = 0 \tag{65}$$

and

$$xU_{p-1}(t) = \omega U_{p-2}(t), \tag{66}$$

where $T_n(t)$ and $U_n(t)$ are the Chebyshev polynomials of the first and second kind, respectively, of degree n (see e.g. [16]). Since $\omega \neq 0$, by dividing (65) by ω^p and using (66), we get

$$U_{p-2}^p(t)T_p(t) - U_{p-2}^{p-1}(t)U_{p-1}(t)T_{p-1}(t) = \frac{1-\omega}{\omega p} U_{p-1}^p(t),$$

or

$$U_{p-2}^{p-1}(t)\left[U_{p-2}(t)T_p(t) - U_{p-1}(t)T_{p-1}(t)\right] = \frac{1-\omega}{\omega^p} U_{p-1}^p(t).$$

By using the trigonometric forms of the quantities inside the brackets in the previous equation, one easily establishes that

$$U_{p-2}(t)T_p(t) - U_{p-1}(t)T_{p-1}(t) = -1.$$

We therefore get

$$U_{p-1}^p(t) = -\frac{\omega^p}{1-\omega} U_{p-2}^{p-1}(t),$$

which will be immediately recognized as Equation (34). Note also that Equation (66) is equivalent to

$$x^p = \omega^p \frac{U_{p-2}^p(t)}{U_{p-1}^p(t)},$$

or by using (34), to

$$x^p = -(1-\omega)U_{p-2}(t),$$

which is recognized as Equation (35), with $x = \gamma(\omega)$.

We have effectively shown that the $p-1$ complex roots $\lambda = xe^{j\theta}$ of $f_{\omega,1}(\lambda)$ are $xe^{j\cos^{-1}t}$, with t equal to a solution of (34) and the corresponding x given by (35). Observe also that, when a root of $f_{\omega,1}(\lambda)$ is $xe^{j\theta} \equiv xe^{j\cos^{-1}t}$ (where $t = \cos\theta$), its complex conjugate is $xe^{-j\theta} = -xe^{j(\pi-\theta)} = -xe^{\cos^{-1}(-t)}$.

We now investigate the roots of (34) in terms of the "dual" variable $\theta \equiv \cos^{-1}t$. Since the right hand side of (34) is always positive (recall that $\omega < 0$ and that p is odd), the roots t must be such that $U_{p-1}(t) > 0$. Since

$$U_{p-1}(t) \equiv U_{p-1}(\theta) = \frac{\sin p\theta}{\sin\theta},$$

we must have

$$\theta \in \left(\frac{2k\pi}{p}, \frac{2k+1}{p}\pi\right), \qquad k = 0, \ldots, \frac{p-1}{2}.$$

Given also that

$$\frac{2k\pi}{p-1} \in \left(\frac{2k\pi}{p}, \frac{2k+1}{p}\pi\right), \qquad k = 1, \ldots, \frac{p-3}{2},$$

we can see that the function

$$G(t) \equiv G(\theta) \equiv U_{p-1}^{p}(\theta) + \frac{\omega^p}{1-\omega} U_{p-2}^{p-1}(\theta)$$

has the property

$$G\left(\frac{2k\pi}{p}\right) = \frac{\omega^p}{1-\omega} U_{p-2}^{p-1}\left(\frac{2k\pi}{p}\right) < 0, \qquad k = 1, \ldots, \frac{p-3}{2}.$$

Similarly,

$$G\left(\frac{2k+1}{p}\pi\right) < 0, \qquad k = 1, \ldots, \frac{p-3}{2}.$$

Finally, since all $2k\pi/(p-1)$ for $k = 1, \ldots, (p-3)/2$ belong to intervals where $U_{p-1}(\theta)$ is positive, we get that

$$G\left(\frac{2k\pi}{p-1}\right)=U_{p-1}^{p}\left(\frac{2k\pi}{p-1}\right)>0,\qquad k=1,\ldots,\frac{p-3}{2}.$$

We therefore conclude that, since $G(t)$ is a continuous function, Equation (34) has at least $p-3$ real solutions in the intervals

$$\left(\cos\frac{2k\pi}{p-1},\cos\frac{2k\pi}{p}\right),\quad\left(\cos\frac{2k+1}{p}\pi,\cos\frac{2k\pi}{p-1}\right),\qquad k=1,\ldots,\frac{p-3}{2}.$$

Also, since for $\theta=0$ we have $U_{p-1}(0)=p$ and $U_{p-2}(0)=p-1$, we get

$$\begin{aligned}G(0)&=p^{p}+\frac{\omega^{p}}{1-\omega}(p-1)^{p-1}\\&=\frac{p^{p}}{(p-1)(1-\omega)}\left[(p-1)(1-\omega)+\left(\omega\frac{p-1}{p}\right)^{p}\right].\end{aligned}$$

We know that (36) holds; therefore we get $G(0)>0$. Also

$$G\left(\frac{\pi}{p}\right)=\frac{\omega^{p}}{1-\omega}U_{p-2}^{p-1}\left(\frac{\pi}{p}\right)<0.$$

These two last relations reveal that Equation (34) has, at least, another root in $(\cos(\pi/p),1)$. Similarly one finds that there is one more root in $(-1,\cos(\pi-\pi/p))$.

Since the solutions of (34) mentioned explicitly above are $p-1$, and since each solution of (34) corresponds to a complex root of $f_{\omega,1}(\lambda)$, these are the *only* solutions of (34).

The last thing that remains to be shown is that the root $\theta^*\in(0,\pi/p)$ [equivalently, the root $t^*=\cos\theta^*\in(\cos(\pi/p),1)$] maximizes

$$|x(t)|\equiv|x(\theta)|=-(1-\omega)U_{p-2}(\theta). \tag{67}$$

Since we know that for any root $\theta(t)$ of (34) there is a conjugate root $\pi - \theta(-t)$, so that

$$x(t) = -(1-\omega)U_{p-2}(t) = -x(-t),$$

we only need to compare the roots $0 < \theta < \pi/2$ with θ^*.

We first treat the roots

$$\phi_m \in \left(\frac{2m\pi}{p}, \frac{2m\pi}{p-1}\right), \qquad m \leqslant \left\lfloor \frac{p-1}{4} \right\rfloor.$$

Obviously, $2m\pi - 2m\pi/p < (p-1)\phi_m < 2m\pi$; hence $\sin(p-1)\phi_m > -\sin(2m\pi/p)$. We also have $\sin(2m\pi/p) < \sin\phi_m$. It is now immediate that

$$|U_{p-2}(\phi_m)| = -\frac{\sin(p-1)\phi_m}{\sin\phi_m} < \frac{\sin(2m\pi/p)}{\sin\phi_m} < 1.$$

Note that, since $\theta^* \in (0, \pi/p)$, we have $U_{p-2}(\theta^*) > 1$, so that $|x(\theta^*)| > |x(\phi_m)|$ for $m = 1, \ldots, \lfloor (p-1)/4 \rfloor$.

Consider now the roots

$$\theta_k \in \left(\frac{2k\pi}{p-1}, \frac{2k+1}{p}\pi\right), \qquad k = 0, \ldots, \left\lfloor \frac{p-2}{4} \right\rfloor.$$

Notice that $\theta_0 = \theta^*$. If $p = 3$, the only root to consider is θ^*, so that there is nothing to prove. For greater p, we will now show that $|x(\theta_k)|$ is a strictly decreasing sequence of index k, something that will convince us that indeed $\theta_0 = \theta^*$ maximizes (67). To start, note that

$$2k\pi < (p-1)\theta_k < 2k\pi + \pi - \frac{2k+1}{p}\pi,$$

so that $U_{p-2}(\theta_k) > 0$; therefore $x(\theta_k) < 0$. Since the corresponding root of $f_{\omega,1}(\lambda)$ is $\lambda_k = x(\theta_k)e^{j\theta_k}$, the polar form of λ_k is $|x(\theta_k)|e^{j(\pi-\theta_k)}$. Let now two such roots of $f_{\omega,1}(\lambda)$ be λ_i and λ_j, with $0<i<j\lfloor(p-2)/4\rfloor$ (so that $0<\theta_i<\theta_j<\pi/2$). Recall that $f_{\omega,1}(\lambda_i)=f_{\omega,1}(\lambda_j)=0$. This gives us

$$\frac{\lambda_j-\omega}{\lambda_i-\omega}=\left(\frac{\lambda_i}{\lambda_j}\right)^p$$

and, by taking absolute values and doing some algebra,

$$\frac{|\lambda_j|^2+\omega^2-2|\lambda_j|\omega\cos(\pi-\theta_j)}{|\lambda_i|^2+\omega^2-2|\lambda_i|\omega\cos(\pi-\theta_i)}=\left(\frac{|\lambda_i|}{|\lambda_j|}\right)^{2p}.$$

Assume now that $|\lambda_j|=|x(\theta_j)|\geqslant|x(\theta_i)|=|\lambda_i|$. The relation above gives

$$\frac{|\lambda_j|^2+\omega^2+2|\lambda_j|\omega\cos\theta_j}{|\lambda_i|^2+\omega^2+2|\lambda_i|\omega\cos\theta_i}\leqslant 1.$$

By recalling that $0<\cos\theta_j<\cos\theta_i$ and that $\omega<0$, we get a contradiction.

We have now established that the root θ^* maximizes (67). The proof of Theorem 4.2 is now complete.

APPENDIX C. PROOF OF THEOREM 4.3

Recall that, from Theorem 3.2, for each root λ of $f_{\omega,1}(\lambda)$, the eigenvector of $\mathscr{L}_\omega$ associated with λ^p is

$$\boldsymbol{\psi}'=\left(\boldsymbol{\psi}_1',\boldsymbol{\psi}_2',\dots,\boldsymbol{\psi}_p'\right)^T=\left(\boldsymbol{\psi}_1,\lambda\boldsymbol{\psi}_2,\dots,\lambda^{p-1}\boldsymbol{\psi}_p\right)^T,$$

where $(\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_p)^T$ is the Perron eigenvector of J. Let now the principal vector associated with the dominant eigenvalue of $\mathscr{L}_\omega$, when $\omega=p/(p-1)$ or (for odd p) $\omega=\omega^*$, be $\boldsymbol{\varphi}$. We will show that

$$\boldsymbol{\varphi}=\boldsymbol{\psi}'+\mathbf{z}, \tag{68}$$

where

$$\mathbf{z}=\left(0,\frac{\lambda^{-(p-1)}}{p}\boldsymbol{\psi}_2,\dots,\frac{i-1}{p}\lambda^{-(p-i+1)}\boldsymbol{\psi}_i,\dots,\frac{p-1}{p}\lambda^{-1}\boldsymbol{\psi}_p\right), \tag{69}$$

which, by using Theorem 3.2, will assure us that the subvectors of $\boldsymbol{\varphi}$ are themselves parallel to the corresponding subvectors of $\boldsymbol{\psi}$. By definition of the principal vector, we just need to show that

$$(\mathscr{L}_\omega-\lambda^p I)\boldsymbol{\varphi}=\boldsymbol{\psi}',$$

or, equivalently,

$$\mathscr{L}_\omega\boldsymbol{\varphi}=\lambda^p\boldsymbol{\varphi}+\boldsymbol{\psi}'.$$

By substituting from (68) and using the fact that $\boldsymbol{\psi}'$ is an eigenvector of $\mathscr{L}_\omega$ associated with the eigenvalue λ^p, the relation to be proved becomes

$$\mathscr{L}_\omega\mathbf{z}=\lambda^p\mathbf{z}+\boldsymbol{\psi}'$$

Call the right hand side $\mathbf{t}$. From the form of $\mathbf{z}$, $\mathbf{t}_1=\boldsymbol{\psi}_1'$ and

$$\mathbf{t}_i=\lambda^{i-1}\left(\frac{i-1}{p}+1\right)\boldsymbol{\psi}_i \qquad \text{for} \quad i=2,\dots,p.$$

The matrix equation now becomes

$$\left[(1-\omega)I+\omega D^{-1}U\right]\mathbf{z}=(I-\omega D^{-1}L)\mathbf{t},$$

or, by using (17) and (7),

$$(1-\omega)\mathbf{z}_1+\omega B_1\mathbf{z}_p=\mathbf{t}_1=\boldsymbol{\psi}_1 \tag{70}$$

and

$$(1-\omega)\mathbf{z}_i=\mathbf{t}_i-\omega B_i\mathbf{t}_{i-1}=\lambda^{i-1}\left(\frac{i-1}{p}+1\right)\boldsymbol{\psi}_i-\omega\lambda^{i-2}\left(\frac{i-2}{p}+1\right)B_i\boldsymbol{\psi}_{i-1}$$

$$=\lambda^{i-1}\left(\frac{i-1}{p}+1\right)\boldsymbol{\psi}_i-\omega\lambda^{i-2}\left(\frac{i-2}{p}+1\right)\boldsymbol{\psi}_i. \tag{71}$$

We now verify Equations (70) and (71). By using (69), Equation (70) becomes

$$\frac{p-1}{p}\omega\lambda^{-1}B_1\boldsymbol{\psi}_p = \boldsymbol{\psi}_1,$$

and by the fact $B_1\psi_p = \psi_1$, we get

$$\frac{p-1}{p}\omega\lambda^{-1}\boldsymbol{\psi}_1 = \boldsymbol{\psi}_1.$$

In the case of $\omega = p/(p-1)$, $\lambda = 1$, and in the case of $\omega = \omega^*$, $\lambda = \gamma(\omega^*)$. In both cases,

$$\lambda = \frac{p-1}{p}\omega,$$

so Equation (70) is seen to hold. Equation (71) holds iff

$$(1-\omega)\frac{i-1}{p} = \left(\frac{i-1}{p}+1\right)\lambda^p - \omega\left(\frac{i-2}{p}+1\right)\lambda^{p-1},$$

or, equivalently, iff

$$\frac{i-1}{p}\left[\lambda^p-(1-\omega)\right] = \frac{i-2}{p}\omega\lambda^{p-1} + \omega\lambda^{p-1} - \lambda^p.$$

Using the fact that $f_{\omega,1}(\lambda) = 0$, after some algebraic manipulation, we get

$$\lambda = \frac{p-1}{p}\omega.$$

As we have already seen, this holds for either ω-value. The proof is now complete.

APPENDIX D. PROOFS OF LEMMAS 6.1, 6.2, AND 6.3

Proof of Lemma 6.1. From Theorems 4.1 and 4.2 one immediately sees that, for the range of ω-values considered, $\gamma(\omega)$ is indeed a function of ω.

From the fact $f_{\omega,1}(\gamma(\omega)) = \gamma^p(\omega) - \omega\gamma^{p-1}(\omega) - (1-\omega) = 0$, by differentiating and solving for $\gamma'(\omega)$, one obtains

$$\gamma'(\omega) = \frac{\gamma^{p-1}(\omega) - 1}{p\gamma^{p-2}(\omega)\left(\gamma(\omega) - \dfrac{p-1}{p}\omega\right)}.$$

If $\omega > p/(p-1)$, we have

$$\gamma(\omega) > \frac{p-1}{p}\omega > 1;$$

therefore $\gamma'(\omega) > 0$. For $\omega < \omega^*$ (case of odd p) we have

$$\gamma(\omega) < \frac{p-1}{p}\omega < -1,$$

and by recalling that $p-1$ is even, we again conclude that $\gamma'(\omega) > 0$. ∎

Proof of Lemma 6.2. We have

$$\left(\frac{\gamma(\omega)}{\omega - 1}\right)' = \frac{p\gamma^{p-1}(\omega)\gamma'(\omega)(\omega-1) - \gamma^p(\omega)}{(\omega-1)^2}.$$

By using the expression for $\gamma'(\omega)$ from the previous proof, after some algebraic manipulation and by using the fact that $f_{\omega,1}(\gamma(\omega)) = 0$, we get

$$\left(\frac{\gamma(\omega)}{\omega - 1}\right)' = \frac{\gamma^p(\omega)}{(\omega-1)^2}\,\frac{\dfrac{p-1}{p}\omega - 1}{\gamma(\omega) - \dfrac{p-1}{p}\omega}.$$

In the case of $\omega > p/(p-1)$, we have $\gamma(\omega) > \omega(p-1)/p > 1$; therefore the derivatives is positive and the function increasing. If $\omega < \omega^* < -p/(p-1)$ (for odd p), then $\gamma(\omega) < \omega(p-1)/p < -1$, so the derivative is now negative and the function decreasing, as was claimed.

Proceeding to the function $\gamma(\omega)/\omega$, using the same technique as before, we determine that

$$\left(\frac{\gamma(\omega)}{\omega}\right)' = (p-1)\frac{\omega - \dfrac{p}{p-1}}{p\omega^2\gamma^{p-2}(\omega)\left(\gamma(\omega) - \omega\dfrac{p-1}{p}\right)}.$$

Using the same arguments as before, one sees that $\gamma(\omega)/\omega$ is increasing for $\omega > p/(p-1)$ and decreasing for $\omega < \omega^*$. ■

Proof of Lemma 6.3. By using the trigonometric form of $U_{p-1}(t)$ and $U_{p-2}(t)$, with $\theta = \cos^{-1} t$, $\theta \in (0, \pi/p)$, we have

$$\left(\frac{U_{p-1}(t)}{U_{p-2}(t)}\right)' = \frac{d}{d\theta}\left(\frac{\sin p\theta}{\sin(p-1)\theta}\right)\cdot\frac{d\theta}{dt}$$

$$= -\frac{p\cos p\theta\sin(p-1)\theta - (p-1)\cos(p-1)\theta\sin p\theta}{\sin^2(p-1)\theta\sin\theta}$$

Proving that the numerator of the fraction above is negative will yield the desired result. Since $\sin p\theta, \sin(p-1)\theta > 0$, the relation to be proved is equivalent to

$$\cot p\theta < \frac{p-1}{p}\cot(p-1)\theta. \tag{72}$$

If $(p-1)\theta < \pi/2 < p\theta$, then obviously

$$\cot p\theta < 0 < \frac{p-1}{p}\cot(p-1)\theta,$$

and the claim holds. Otherwise, $\cot p\theta$ and $\cot(p-1)\theta$ have the same sign, and (72) is equivalent to

$$\frac{p}{p-1}\tan(p-1)\theta < \tan p\theta,$$

or to

$$\varphi(\theta) \equiv \frac{p}{p-1} \tan(p-1)\theta - \tan p\theta < 0.$$

It is easy to establish that $\varphi'(\theta) < 0$ for $0 < (p-1)\theta < p\theta < \pi/2$ and $\varphi'(\theta) > 0$ for $\pi/2 < (p-1)\theta < p\theta < \pi$, so that in the first case $\varphi(\theta) < \varphi(0) = 0$, and in the second case

$$\varphi(\theta) < \varphi\left(\frac{\pi}{p}\right) = -\frac{p-1}{p} \tan \frac{\pi}{p} < 0.$$

We have therefore proved the lemma for any $\theta \in (0, \pi/p)$—equivalently, for all $t \in (\cos(\pi/p), 1)$.

REFERENCES

1 T. M. Altiok and S. Stidham, A note on transfer lines with unreliable machines, random processing times and finite buffers, *IIE Trans.*, June 1982, pp. 125–127.

2 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.

3 Y. T. Chen, Iterative Methods for Least Squares Problems, Ph.D. Thesis, Dept. of Computer Science, Univ. of Waterloo, Waterloo, Ontario, Canada, 1975.

4 P. J. Courtois, *Decomposability: Queueing and Computer Systems Applications*, Academic, New York, 1977.

5 P. J. Courtois and P. Semal, Block iterative algorithms for stochastic matrices, *Linear Algebra Appl.* 70:59–76 (1986).

6 M. Eiermann, W. Niethammer, and A. Ruttan, Optimal successive overrelaxation iterative methods for p-cyclic matrices, *Numer. Math.*, 57:593–606 (1990).

7 M. Gutknecht, W. Niethammer, and R. S. Varga, K-step iterative methods for solving nonlinear systems of equations, *Numer. Math.* 48:699–712 (1986).

8 A. Hadjidimos, On the optimization of the classical iterative schemes for the solution of complex singular linear systems, *SIAM J. Algebraic Discrete Methods* 6(4):555–566 (1985).

9 C. D. Meyer, Stochastic complementation, uncoupling Markov chains and the theory of NCD systems, *SIAM Rev.*, 1989, pp. 240–272.

10 Debasis Mitra and P. Tsoukas, Relaxations for the numerical solutions of some stochastic models, *Stochastic Models* 4, No. 3 (Nov. 1988).

11 W. Niethammer, Relaxation bei Matrizen mit der Eigenschaft "A," *Z. Angew. Math. Mech.* 44:T49–T52 (1964).

12 W. Niethammer, J. de Pillis, and R. S. Varga, Convergence of block iterative methods applied to sparse least squares problems, *Linear Algebra Appl.* 58:327–341 (1984).

13 H. Perros and A. Altiok (Eds.), *Queueing Networks with Blocking*, North Holland, Amsterdam, 1989.
14 D. J. Pierce, A. Hadjidimos, and R. J. Plemmons, Optimality relationships for p-cyclic SOR, *Numer. Math.* 56:635–643 (1990).
15 R. J. Plemmons, Adjustments by least squares in geodesy using block iterative methods for sparse matrices, in *Proceedings of the Annual U.S Army Conference on Numerical Analysis and Computation*, 1979, pp. 151–186.
16 T. J. Rivlin, *The Chebyshev Polynomials*, Wiley, New York, 1974.
17 W. J. Stewart, Personal communication.
18 R. S. Varga, p-cyclic matrices: A generalization of the Young-Frankel successive overrelaxation scheme, *Pacific J. Math.* 9:617–628 (1959).
19 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
20 P. Wild and W. Niethammer, Over- and underrelaxation for linear systems with weakly cyclic Jacobi matrices of index p, *Linear Algebra Appl.* 91:29–52 (1987).
21 D. M. Young, Iterative methods for solving partial differential equations of elliptic type, *Trans. Amer. Math. Soc.* 76:92–111 (1954).
22 S. Galanis and A. Hadjidimos, Best cyclic repartitioning for optimal SOR convergence, *SIAM J. Matrix Anal. Appl.*, to appear.

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